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Balancing of Engines

STEAM, GAS, AND PETROL

Archibald Sharp

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STEAM, GAS, AND PETROL

AN ELEMENTARY TEXT-BOOK, USING
PRINCIPALLY GRAPHICAL METHODS

*FOR THE USE OF STUDENTS, DRAUGHTSMEN,
DESIGNERS, AND BUYERS OF ENGINES*

WITH NUMEROUS TABLES AND DIAGRAMS

BY

ARCHIBALD SHARP

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MEMBER OF THE INSTITUTION OF AUTOMOBILE ENGINEERS;
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PREFACE

THE development of the motor-car engine, and the successful installation of steam turbines for land and marine use, have been the means of directing more attention to the subject of engine balancing. The smooth running of an engine depends mainly on two factors: a more or less uniform torque on the crank-shaft, and good balance of the inertia forces of the engine. The latter forms the subject-matter of the present work; the former cannot be left without any mention, a chapter is therefore devoted to it.

With the exception of a few analytical investigations, the method of treatment is graphical. In fact, the engineering student will recognize many geometrical methods with which he is familiar as applied to Statics of Structures.

The only standard work on Balancing of Engines hitherto available is by Professor W. A. Dalby, to which the Author is greatly indebted. In dealing with longitudinal couples, the Author has generally adopted a purely graphical method, instead of the semi-graphical, semi-arithmetical method used by Professor Dalby. Both methods have their own special merits.

In discussing the number of variables to specify n masses, and for the self-balancing of a multiple-crank engine, the reader may at first sight think the results are different from those given by Professor Dalby. The difference, however, is only apparent, not real. By a slightly different assumption at the outset, the discussion of the problem seems to be simplified in some cases.

Some of the subject-matter, as far as the Author is aware, is published for the first time. Part of the subject-matter

of Chapter VIII. has been taken from a paper by the Author, published in the *Minutes of Proceedings of the Institution of Civil Engineers,* vol. clxviii. The expressions for the kinetic energy of the piston and connecting-rod in terms of the crank-shaft angle are new. The method of estimating the resultant torque on the crank-shaft for all speeds and loads, explained in Chapter X., should materially reduce the draughtsman's labour.

At the time of writing, the petrol engine for marine purposes, principally for pleasure or racing launches, has been well established. It seems probable that gas or oil engines of large power will, in the near future, be used in the Navy and mercantile marine service. Some space is, therefore, devoted to the discussion of some types of engine, non-existent up to the present, but which may be desirable types if gas engines of 10,000 horse-power and upwards be required on board a ship.

To make the work as useful as possible to students, a series of exercises, with answers, in some cases, are appended at the end of most of the chapters, some of these being taken from the papers set at the Qualifying Examinations of the Institution of Civil Engineers.

The Synopsis at the end of the volume presents, in a form convenient for easy reference, the inertia properties of most of the types of engines in actual use. It is most improbable that errors in calculation have been entirely avoided. Any corrections will be gladly received by the Author. The Author is indebted to Mr. G. A. Burls, M.Inst.C.E., for reading the proof-sheets.

231, Strand,
London, W.C.

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CHAPTER I

PRIMARY PHENOMENA OF MOTION AND FORCE

THE fundamental conceptions of *time, space*, and *mass* are assumed to be known by the student, as well as the various units employed for their measurement. It seems desirable, however, to give a few notes on the derived physical quantities with which the engineer has to deal, before going on with the special subject-matter of this volume.

(1) **Speed.**—The speed of a body is the rate at which it passes through space. Speed may be *uniform* or *variable*. With uniform speed the body passes through equal spaces in equal times, with variable speed the spaces passed through in equal time intervals are unequal. The unit of speed is the rate of passing through unit space in unit time. $v = s/t$.

(2) **Velocity involves Direction as well as Speed.**—If the direction and speed are known, the velocity is completely defined. Graphically, velocity may be completely represented by a straight line of finite length, the length of the line representing to any convenient scale the speed of the body, the direction of the line representing the direction in which the body is moving. In *linear* velocity, the space passed through by the body is along a line, which may be either straight or curved. Thus, on a crank-shaft revolving at the rate of one revolution per second, a point one foot from the axis of the shaft will in one second describe the circumference of a circle of one foot radius, *i.e.* its linear

speed is 2π foot per second $= 6{\cdot}2832$ feet per second. The direction of its linear motion is changing from instant to instant; one element of its linear velocity is continually changing, and therefore, although the *speed* is uniform, the *velocity* is varying.

(3) **Momentum.**—The momentum of a moving body is the produce of its mass and velocity, mv.

(4) **Angular Speed.**—The crank-shaft turning with a uniform speed about its axis, every point on it describes the same *angle* in the same interval of time. The angular speed of a rigid body rotating about a fixed axis is the same as the angular speed relative to that axis of any point on the rigid body. In rotary motion, therefore, the element of space is an *angle*, and the unit of angular speed is unit angle described in unit time. The phrase "60 revolutions per minute" expresses the angular speed of a shaft quite definitely, the unit angle being implicitly taken as the angle swept through in one revolution, *i.e.* 360 degrees, or 2π radians. The radian is the unit angle of circular measure; *i.e.* the angle subtended by a circular arc of length equal to its radius.

If N be the number of revolutions per minute, ω the angular speed in radians per second, then evidently,

$$2\pi \mathrm{N} = 60\omega$$

or,

$$\omega = \frac{\pi \mathrm{N}}{30} = 0{\cdot}1047\mathrm{N} \quad . \quad . \quad . \quad . \quad (1)$$

If v be the linear speed of a point at radius r on a rigid body rotating at angular speed ω, then

$$v = \omega r, \text{ or } \omega = v/r \quad . \quad . \quad . \quad . \quad (2)$$

Angular Velocity.—If the direction of the axis of rotation is variable, to specify the motion completely, the *direction* of the axis, as well as the speed of rotation, must be given. Thus the motion of the flywheel of a gyroscope at any instant can be completely represented graphically in the same way as the *linear velocity* of a body.

(5) **Acceleration.**—Acceleration is rate of change of

velocity. The speed may change while the direction of the motion is unaltered (as in a body falling freely under the influence of gravity) in which case the acceleration is in the same direction as the velocity. Or, the speed of the body may remain uniform while the direction of its motion is continually changing (as in a body moving in a circular path with uniform speed) in which case the acceleration is at each instant at right angles to the direction of the velocity. If both speed and direction are changing, both kinds of acceleration are present. The acceleration along the path is called "tangential" acceleration, that at right angles "normal" or "radial." The resultant of the tangential and radial accelerations is called the total acceleration.

Acceleration, involving direction, as well as magnitude, can be completely represented graphically by a finite straight line.

The unit of acceleration is unit of speed added per unit of time.

If the velocity v is uniformly added in t units of time, v/t units of velocity are added in one unit of time. Therefore

$$a = v/t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Angular Acceleration.—If the angular speed of a shaft varies, it is said to have angular acceleration. If the angular speed ω is uniformly added in t units of time, ω/t units are added in one unit of time. Therefore the angular acceleration, which is the rate of increase of angular speed, is $\phi = \omega/t$.

(6) **Force and Mass.**—Force is that which tends to produce, alter, or vary motion. In rational mechanics the unit of force (absolute unit of force) is that force which, acting on unit of mass, produces unit acceleration.

To completely specify a force we require to know its magnitude, direction, and line along which it acts. Therefore, it can be completely represented graphically by a finite line, in a definite direction, and in a definite position.

To produce unit acceleration on a body of m units of mass would therefore require a force of m units as above defined. To produce a units of acceleration on a body of

unit mass would require a force of a units. To produce a units of acceleration on a body of m units of mass would require a force of ma units. Therefore

$$f = ma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Since a is the rate of increase of velocity, *i.e.* $a = v/t$, equation (4) may be written

$$ft = mv \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

v being the velocity *added* in the time t. The product ft is called the *impulse* of the force; the product mv is the momentum *added*. Therefore, equation (5) may be stated,

Impulse = momentum added.

(7) **Work, Energy, Power.**—*Work* is measured by the product of the acting *force* and the *distance* moved in the direction of the force. In lifting a weight, w lbs., through a vertical height h ft., the work done is wh foot-lbs. It is immaterial what mechanical appliances be used, whether wedge, inclined plane, pulleys. *Work* may also be expressed as the product of the *component of the force* in the direction of motion and the *distance* moved. Thus, in lifting 100 lbs. weight 2 feet high by means of an inclined plane of 1 in 10, the length of the slope would be 20 ft. and the force to be exerted parallel to the inclined plane 10 lbs.; the work done $= 10 \times 20 = 200$ ft.-lbs.

If the direction of motion is at right angles to the direction of the force, no *work* is done. This, neglecting friction, is the case of moving weights on horizontal rails, forces on bearings, centrifugal force; the said forces may alter the direction of motion at any instant, but they do no work.

If a body free to move is acted on by a force, the work done by the force is stored up as *kinetic energy* of the moving body. It is shown in elementary text-books of mechanics that the kinetic energy of a body of mass m moving with linear speed v is $mv^2/2$ absolute units of work or energy, or $mv^2/2g$ gravitation units, (foot-lbs., if the foot and lb. weight are the units of length and force respectively).

Power is rate of doing work. The *horse-power* is the

engineer's unit of power, and is defined as that of an agent doing work at the rate of 33,000 ft.-lbs. per minute, or 550 ft.-lbs. per second.

(8) **Laws of Motion.**—Our conceptions of *mass* and *force* are intimately connected one with the other. *Mass, i.e.* quantity of matter, we estimate by its "inertia" or resistance to being disturbed from its existing state of rest or motion, when acted on by force. *Force* we estimate by its effect in altering the existing state of rest or motion of a mass. Thus, we can form no idea of mass without having an idea of force, and conversely. The fundamental laws concerning mass, force, and motion are concisely expressed in Newton's three "Laws of motion," which should be regarded as generalized statements regarding the phenomena of motion and force, as exhibited in all parts of the universe of which we have any knowledge.

I. Every body continues in its state of rest or of uniform motion in a straight line, unless compelled by impressed force to change that state.

II. Change of motion is proportional to the force applied, and takes place in the direction in which the force acts.

In the modern nomenclature of mechanics, Newton's second law of motion may be expressed thus: Rate of change of momentum is proportional to the force applied; and—

If mv be the momentum added uniformly in t units of time, mv/t is the rate of change of momentum. But $v/t = a$. Therefore, Newton's second law of motion is an enunciation of the dynamical equation $f = ma$.

III. Action and reaction are equal and opposite.

From the first law, it follows that every mass moving in a circle must be continuously acted on by an applied force, since the velocity of the body is being continuously changed in direction. In the case of a shaft, wheel or pulley, turning about its axis, each element is acted on by force. As we shall see later, these forces vary as the square of the speed of the shaft, and at high speeds of rotation the rotating body is subjected to considerable stresses (centrifugal stresses) due to the speed.

The second law, as expressed by the formula $f = ma$, is of fundamental importance in all dynamical problems. f in the formula is to be regarded as the resultant of all the forces acting on the body. If a body is at rest or moving with constant velocity in a straight line, a is zero, and therefore f is zero; *i.e.* all the forces acting on the body are in equilibrium among themselves.

The third law expresses the fact that no single isolated force can exist. The forces the engineer has to deal with are caused by the mutual contact of bodies—solid, liquid, or gaseous. If a body A exerts a force F on a body B, the reaction of B on A is a force $-$ F. The pressure of a shaft on its bearing is equal and opposite to the reaction of the bearing on the shaft. Care must always be taken that the correct direction be shown for each force in any mechanical problem the student is dealing with.

(9) **Straining Actions, Stress.**—In a structure at rest, or moving with no acceleration, yet subjected to external forces, each portion has zero resultant force acting on it. A long bar, for example, acted on by external forces at its ends, must have the forces applied at the ends equal and opposite. If the forces applied at the ends are directed towards each other the bar is said to be in *compression;* if directed away from each other, the bar is said to be in *tension.*

If in a structure one portion is in tension, the other portion must be in compression, since the action and reaction are mutual. Thus, in an engine, if the gases in the cylinder produce a state of compression on the piston-rod, connecting-rod, and crank, the opposite reaction is a tension on the cylinder and frame, between the cylinder head and crank-shaft bearings.

(10) **Dimensions and Nomenclature of Units.**—If the three fundamental units of space, time, and mass be denoted by (m), (s), and (t) respectively, the unit of speed (v) is $(s)/(t)$. The unit of acceleration is

$$(a) = (v)/(t) = (s)/(t)^2.$$

The unit of force is

$$(f) = (m)(a) = (m)(s)/(t)^2.$$

Thus, in the pound, foot, second system of units, the unit of speed is one foot per second, written 1 ft./sec. The unit of acceleration is 1 ft./sec.^{2} The absolute unit of force is 1 lb. ft./sec.^{2}, sometimes called the *poundal*.

The engineer's unit of force, or the *gravitational* unit of force, is the *weight* of unit mass, that is, the force with which the earth attracts unit mass. If g is the acceleration of a body falling freely under the influence of gravity, then considering unit mass to fall freely under gravity, it is acted on by a force equal to one *gravitation* unit, which produces g units of acceleration. The gravitation unit of force is therefore equal to g absolute units

$$g = 32{\cdot}2 \text{ ft./sec.}^2,$$

therefore

$$1 \text{ lb. weight} = 32{\cdot}2 \text{ lb.-ft./sec.}^2 = 32{\cdot}2 \text{ poundals.}$$

If a certain force is expressed as f absolute units or F gravitation units, then

$$\mathrm{F} = f/g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the gramme, centimetre, second system of units, the absolute unit of force (1 grm. cm./sec.^{2}) is called a dyne. $g = 981$ cm./sec.^{2}. Therefore, 1 gramme weight = 981 dynes.

(11) **Vectors and Vector Quantities.**—Velocity, momentum, acceleration, a force acting at a point, can each be represented graphically by the same geometrical quantity, a finite straight line whose length represents to any convenient scale the magnitude, and whose direction is the same as that of the velocity, acceleration, or force represented. The name *vector* is given to this geometric quantity. If any physical quantity can be graphically represented by a vector, it is called a "vector quantity." The laws of combination of vector quantities are the same as those of vectors.

If a finite straight line OA is drawn on the paper, it may represent a step from O to A, or one from A to O, one being in the direction opposite to the other. To indicate definitely

one of the two directions, an arrow-head should be placed on the line. With the usual convention as to + and − signs

$$\text{Vector } \hat{OA} = -\text{vector } \hat{AO}. \quad . \quad . \quad . \quad (7)$$

The nature of vector addition can be simply and clearly shown by considering a series of "displacements," or "steps," a step being a vector quantity. Thus, in Fig. 1, a step from O to A, combined with a step from A to B, is equivalent to a single step from O to B. That is

$$\text{Vector } \hat{OA} + \text{vector } \hat{AB} = \text{vector } \hat{OB}.$$

Similarly

$$\text{Vector } \hat{OD} = \text{vector sum } (\hat{OA} + \hat{AB} + \hat{BC} + \hat{CD}).$$

In vector addition, the order in which the summation is taken does not affect the result. Thus, in Fig. 1, the four

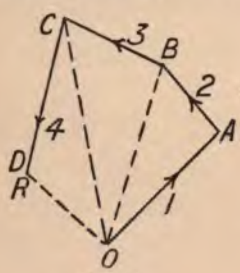

FIG. 1.

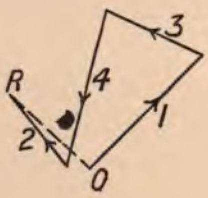

FIG. 2.

FIG. 3.

vectors 1, 2, 3, and 4 are added in the order mentioned; in Fig. 2 the same four vectors are added in the order 1, 3, 4, 2, the vector sum OR being the same in Figs. 1 and 2.

If the vectors 1, 2, 3, and 4 in Fig. 1 represent velocities, or forces acting at a point, the vector sum $\hat{OR}$ will represent the *resultant* velocity or resultant force. It must be carefully noted that in drawing the vector-polygon (Fig. 1 or 2) to obtain the vector sum, the arrow-heads must run in order round the sides of the polygon.

If a fifth vector, $\hat{RO}$, be added to the four vectors $\hat{OA}$, $\hat{AB}$, $\hat{BC}$, $\hat{CD}$ (Fig. 1), the sum of the five vectors, 1, 2, 3, 4, and 5 is zero (Fig. 3). Thus, if the resultant of a number of vector quantities is zero, the vector-polygon must

be closed; conversely, if the vector-polygon is closed, the resultant vector quantity is zero.

(12) **Resolution of Vectors.**—From the above discussion, it is readily seen that any vector $\widehat{OA}$ (Fig. 4) can be resolved into two component vectors along any two directions, OX and OY. From A draw Aa' and Aa'' parallel to OY and OX respectively, meeting OX and OY at a' and a'' respectively. Then

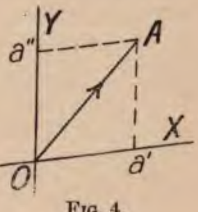

FIG. 4.

$$\text{Vector } \widehat{OA} = \text{vector } \widehat{Oa'} + \text{vector } \widehat{a'A}.$$

But

$$\text{Vector } \widehat{a'A} = \widehat{Oa''}.$$

Therefore

$$\text{Vector } \widehat{OA} = \text{vector } \widehat{Oa'} + \text{vector } \widehat{Oa''}.$$

Resolving the four vectors and the vector sum $\widehat{OR}$ of Fig. 1 along OX and OY, as shown in Fig. 5, it is readily seen that the component $\widehat{Or}$ in the direction OX of the vector sum $\widehat{OR}$ is equal to the sum of the component vectors

$$\widehat{Oa'} + \widehat{a'b} + \widehat{b'c'} + \widehat{c'r'}.$$

Similarly for the components in the direction OY.

Further, comparing Figs. 4 and 5, it is readily seen that if the vector-polygon is closed, the sum of the component vectors in any direction is zero. Conversely, if the sums of the component vectors in two directions are each zero, the vector-polygon is closed.

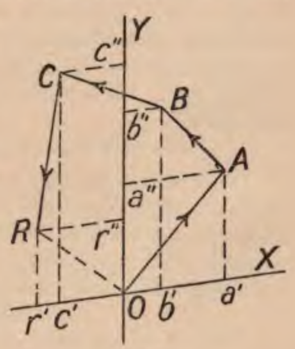

FIG. 5.

EXERCISES.

1. A shaft turns at the rate of 100 revolutions per minute. What is its angular speed in radians per second? *Ans.* 10·47.

2. An engine of 3-ft. stroke runs at 150 revolutions per minute. What is the mean piston speed, and the linear speed of the crank-pin? *Ans.* 15 ft./sec., 23·6 ft./sec.

3. Express in poundals and in lbs. weight the force required to give a mass of a ton an acceleration of 2 ft./sec.^{2}

Ans. 4480 poundals, 140 lbs. weight.

4. A crane in lifting a weight of 5 tons gives it a starting acceleration of $\frac{1}{2}$ ft./sec.^{2} What is the starting pull on the cable?

Ans. 5·078 tons.

5. In example 4, if the acceleration is continued at the same rate until the weight is raised 20 feet, find the total work done, the work done against gravity, and the kinetic energy. *Ans.* 101·56, 100, 1·56 ft.-tons.

6. In example 5, find the speed when the load has been lifted 20 feet. If the lifting is continued at this speed, what horse-power is expended?

Ans. 4·47 ft./sec., 91 H.-P.

7. If g, the acceleration of a body falling freely, is 32·2 ft./sec.,^{2} express its value when a mile and an hour are taken as the units of space and time. *Ans.* 79036 mile/hour.^{2}

8. Find graphically the resultant of vectors 2, 3, 2, 4 units length, drawn respectively, east, north, south-west, and north-west. Find also the components of the resultant vector, in the east and north directions.

Ans. 4·95, 2·21, 4·41.

9. A train is travelling at the rate of 20 mile/hour, and a man, sitting in a corner of a compartment with both windows down, observes a stone pass in a straight line at right angles to the length of the train through both windows. If it appears to the man to have a velocity of 20 ft./sec., with what horizontal velocity must the stone have been thrown? (Inst. C.E., Feb. 1899.) *Ans.* 35·5 ft./sec.

10. A cricket-ball weighing $4\frac{1}{2}$ ozs. is travelling horizontally with a velocity of 30 ft./sec. when it reaches the bat. Supposing the time of the blow in each of three cases to be $\frac{1}{80}$ second, compare the magnitudes of the average force of the blow when the ball is despatched with unaltered speed in directions making angles with the initial direction of (a) 35°, (b) 90°, (c) 180°. Give the direction of the blow in each case. (Inst. C.E., Feb. 1899.)

Ans. 4·74 lbs. at $72\frac{1}{2}$°; 11·27 lbs. at 45°; 15·82 lbs. at 180°.

11. A steam hammer weighs 10 tons, and the steam pressure is 50 lbs./in.^{2}, on the piston of 21 inches diameter. Find (i) the acceleration at which the hammer comes down, (ii) the hammer's velocity after descending 3 feet, (iii) the mean force of the blow if the material being worked is compressed $\frac{1}{2}$ inch. (Inst. C.E., Feb. 1907.)

Ans. 57·1 ft./sec.^{2}, 18·5 ft./sec., 1276 tons.

Note.—A higher steam pressure than 50 lb./in.^{2} is required to lift the hammer.

12. A body is rotating about an axis with an angular velocity of 4 radians per second, and about an axis intersecting the former at an angle of 60° with an angular velocity of 9 radians per second. Find the axis

about which the resultant rotation is taking place, and its amount. (Inst. C.E., Feb. 1900.)

Ans. 11·53 radians/sec., axis at 17·5° to the latter axis.

13. Explain how to determine the relative velocity of two bodies. A is travelling due north at constant speed. When B is due west of A, and at a distance of 21 miles from it, B starts travelling north-east with the same constant speed as A. Determine, graphically or otherwise, the least distance which B attains from A. (Inst. C.E., Feb. 1903.)

Ans. 8·03 miles.

14. Two men put a railway-waggon weighing 5 tons into motion by exerting on it a force of 80 lbs. The resistance of the waggon is 10 lbs. per ton, or altogether 50 lbs. How far will the waggon have moved in one minute? Calculate at what fraction of a horse-power the men are working at 60 seconds after starting. (Inst. C.E., Feb. 1903.)

Ans. 154·3 ft., 0·75 H.-P.

15. State and explain fully Newton's "Third Law of Motion." A 100-lb. shot leaves a gun horizontally with a muzzle velocity of 2000 feet per second. The gun and attachments, which recoil, weigh 4 tons. Find what the resistance must be that the recoil may be taken up in 4 feet, and compare the energy of recoil with the energy of translation of the shot. (Inst. C.E., Feb. 1903.) *Ans.* 7·78 tons; 1 : 90.

CHAPTER II

PRELIMINARY THEOREMS IN MECHANICS

(1) **Body moving in a Circular Path with Uniform Speed.**—Let the uniform linear speed be v, the radius of the circular path (Fig. 1) be r. Let the body be at A at the beginning of a certain interval of time t, and at B at the end of that interval. Then the arc AB is equal to vt. At the beginning and end of the time t, the directions of the motion are AA′ and BB′, respectively at right angles to the radii OA and OB. The velocities of the body when at A and B are represented by the vectors $\hat{ao}$ and $\hat{ob}$, each of length v, respectively parallel to AA′ and BB′. Therefore the velocity added in the time t is represented by the arc ab. If the interval t be taken infinitely small, ab is parallel to AO, that is, the *added* velocity is in the direction of the radius at the instant, and directed *towards* the centre. The *radial* acceleration a is $= ab/t$. Now, OAB and oab are similar figures, therefore

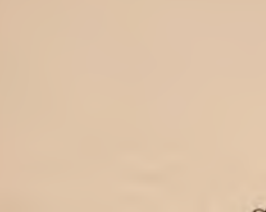

FIG. 1.

$$\frac{\overline{ab}}{\overline{\mathrm{AB}}} = \frac{oa}{\mathrm{OA}}, \quad i.e. \frac{ab}{vt} = \frac{v}{r}, \quad \therefore \frac{\overline{ab}}{t} = \frac{v^2}{r}$$

$$\text{that is, } a = v^2/r \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Substituting $v = \omega r$, (1) may be written

$$a = \omega^2 r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If m is the mass of the body, the force f, in absolute units, directed towards the centre, required to produce the radial acceleration is

$$f = mr\omega^2 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In gravitation units,

$$F = mr\omega^2/g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The above force is a "centre-seeking," or "centripetal" force, which must be exerted on the mass by virtue of the mechanical constraint; the equal and opposite force of reaction exerted by the mass on the part to which it is fastened is the "centrifugal" force.

The radial acceleration, or the radial force, can be represented by a vector rotating with the same angular speed ω.

(2) **Forces acting at a Point.**—Each force can be completely represented by a vector, and therefore the forces can be treated by the methods of secs. 11 and 12, Chap. I.

(3) **Moment of a Force.**—Let F_1 be any force acting on a rigid body, O any point, and p_1 the perpendicular distance from O to the line of action of F_1 (Fig. 2); then the moment of the force F_1 about the point O is the product F_1p_1. If the rigid body is constrained in such a manner that it can only turn about O as an axis, the moment F_1p_1 is a measure of the turning effort of the force F_1. The turning effort or moment P_1p_1 is counter clockwise, the moments of F_2 and F_3 (Fig. 2) are clockwise. If the moment of F_1 be considered positive, those of F_2 and F_3 are negative. If a number of forces act on a body, the moment of their resultant about any axis is equal to the algebraic sum of their moments about that axis.

FIG. 2.

(4) **Couple.**—Let two equal parallel but oppositely directed forces F_1 and F_2 (Fig. 3) act on a rigid body, p being the perpendicular distance between their lines of action. Let O be any point in the plane of F_1 and F_2 at a

perpendicular distance x from the line of action of F_2. Then the algebraic sum of the moments of F_1 and F_2 about the axis O is

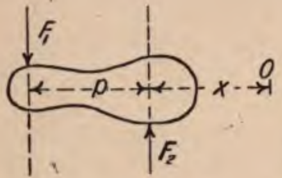

FIG. 3.

$$F_1(x + p) - F_2x = Fp,$$

a result independent of the position of the point O; that is, the algebraic sum of the moments of two equal opposite parallel forces is the same for all points in the plane of the forces.

The two equal and opposite forces constitute a couple, and the product Fp is called the moment of the couple.

(5) **Couple a Vector of Quantity.**—A couple does not tend to produce any motion of translation, but only a turning moment about any axis perpendicular to the plane of the two equal opposite forces. A couple, therefore, is a vector quantity, its magnitude being the moment Fp, and its direction indicated by any line perpendicular to its plane. The couple shown in Fig. 3 being positive, the vector may be drawn with the arrow-head pointing from the plane of paper *towards* the reader; a negative couple, clockwise in the plane of the paper, would be represented by a vector drawn with the arrow-head pointing *away* from the reader. With this convention, the turning effort of the couple, and the direction indicated by the vector arrow-head along the axis have the same relation as the turning and the axial travel of a right-hand screw engaging with a nut.

(6) **Couple of Transference.**—Let F_1 (Fig. 4) be any force acting on a rigid body. Introduce two equal and opposite forces F_2 and F_3 each equal to F_1, and acting in a line parallel to that of F_1 and distant p from it. The two forces F_2 and F_3 neutralize each other, and, therefore, make no alteration in the conditions of the body. The two equal parallel and opposite forces F_1 and F_3 constitute a couple of moment Fp, leaving the force F_2

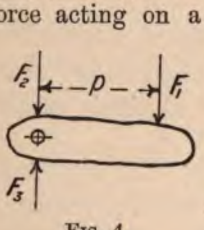

FIG. 4.

parallel to the original force F_1 and distant p from it. Thus, any force F acting on a body may be replaced by an equal parallel force at any distance p, plus a couple of moment Fp; the couple Fp being called the *couple of transference*.

Conversely, the resultant of a force and a couple of moment M, having its axis perpendicular to the force, may be expressed as an equal parallel force F at a distance $p = M/F$ measured from a point in the line of action of the force in a direction at right angles to F and the axis of the couple.

A physical illustration of the above is afforded by screwing up a nut by means of a spanner of length p. The force F, applied at the end of the spanner, is equivalent to a couple Fp tending to turn the spanner and nut, plus a force F_2 which exerts pressure on the nut tending to shear it off the bolt.

(7) **Coplanar Forces acting on a Rigid Body.**—In sect. 6, Chap. I., it has been shown that a force requires for its complete specification three quantities: magnitude, direction, and place or position. The first and second quantities can be represented by a vector, the three quantities by a *localized vector*. From sect. 6, it is evident that any number of given forces are equivalent to a series of equal parallel forces all acting at the same point, which may be chosen anywhere convenient, together with their respective couples of transference. The resultant force at the chosen point is determined in magnitude and position by drawing a vector-polygon, sect. 11 Chap. I.; the resultant couple is determined by algebraic addition. From sect. 6, the resultant force F at the chosen point, and the resultant couple M, are equivalent to an equal parallel force F displaced a distance M/F.

The position of the resultant F of the given forces is more conveniently determined as follows:—Let Fig. 5 represent the position of the given forces P_1, P_2, . . .; Fig. 6 the corresponding force vector-polygon, the resultant F forming the closing side. No difference is made if two equal and opposite forces, acting along the same straight line, be added

to the system. Let a force Q acting along any line a (Fig. 5) be added, Q being represented in the force-polygon (Fig. 6) by $\widehat{Oa}$. The resultant of Q and P_1 is $\widehat{Ob}$ (Fig. 6), and its line of action passes through p_1 (Fig. 5), the point of intersection of Q and P_1. From p_1 draw the line b parallel to Ob (Fig. 6), cutting the line of action of P_2 at p_2. The resultant of Q, P_1 and P_2 is $\widehat{Oc}$ (Fig. 6), and it passes through p_2.

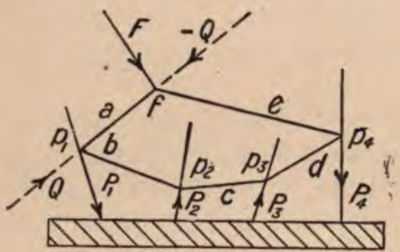

FIG. 5.

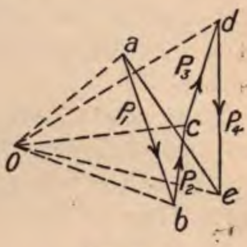

FIG. 6.

From p_2 draw the line c parallel to Oc (Fig. 6). Continuing this process, the resultant of Q, P_1, P_2, . . . P_4 is $\widehat{Oe}$ (Fig. 6), and it passes through the point p_4 (Fig. 5). From p_4 draw the line e parallel to Oe (Fig. 6), cutting a, the line of action of Q at f. At f introduce the force $-$ Q, acting along a in the opposite direction to Q. The resultant of Q, P_1, P_2, . . . $P_4 -$ Q (that is, the resultant of the given forces) is $F = \widehat{ae}$ (Fig. 6), and passes through the point f (Fig. 5).

(8) **Link-Polygon.**—The above construction may be expressed thus: Take any pole O, and from it draw radius vectors to the corners a, b, c . . . e of the force-polygon (Fig. 6). Draw another polygon (which is called the *link-polygon*) having its corners p_1, p_2 . . . on the lines of action of the given forces P_1, P_2 . . . (Fig. 5), and having its sides a, b . . . parallel to the radius vectors Oa, Ob . . . of the force-polygon. The point of intersection f of the first and last sides of the link-polygon determines the position of the resultant force F.

(9) **Conditions of Equilibrium.**—If to the four forces, P_1, P_2, P_3, P_4, in Fig. 5, be added a fifth force P_5, equal and opposite to F, and having the same line of action, the five

forces are in equilibrium. The conditions to be satisfied for a number of forces in equilibrium may therefore be expressed thus: *The force- and link-polygons must each be closed.*

If a closed force-polygon be drawn at random (Fig. 6), and the lines of action (Fig. 5) be taken in random positions, but parallel to the corresponding sides of the force-polygon, in general the link-polygon will not be closed. The resultant force is zero, but the link-polygon being drawn shows two parallel sides which intersect at an infinite distance. A zero force at an infinite distance is equivalent to a *couple*.

(10) **Parallel Forces.**—If the forces are all parallel (Fig. 7), the constructions for the force- and link-polygons

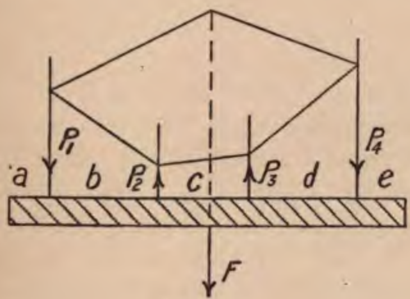

FIG. 7.

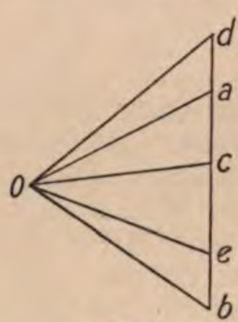

FIG. 8.

are simplified. The sides of the force-polygon all lie in the same straight line (Fig. 8). In this case, which will be of frequent occurrence in discussing the balancing of engines, the notation shown in Figs. 7 and 8 should be adopted. The lines of action of the given forces, P_1, P_2 . . . P_4, subdivide the space round the body into the spaces *a*, *b* . . . *d*, *e*. The force P_1 between the spaces *a* and *b* is represented in the force-polygon (Fig. 8) by the vector $\widehat{ab}$, similarly for the other given forces. The resultant force F is represented by the vector $\widehat{ae}$, and in Fig. 7, F lies between the spaces *a* and *e*. The sides of the link-polygon lying in the spaces *a*, *b*, *c* . . . (Fig. 7) are respectively parallel to the radius vectors O*a*, O*b*, O*c* . . . (Fig. 8).

The resultant force F may be equilibrated by two parallel

forces, F_1 and F_2, acting at any points Q_1 and Q_2 (Fig. 9), which are determined in magnitude by drawing another force-polygon and link-polygon as follows:—The forces F, F_1, and F_2 separate the spaces *a*, *e*, and *q* (Fig. 9). The force F between the spaces *a* and *e* being known, its vector *ae* (Fig. 10) can be marked off, and the radius vectors O*a* and O*e*

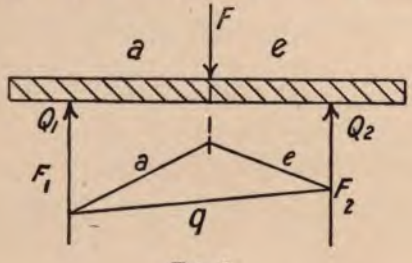

FIG. 9.

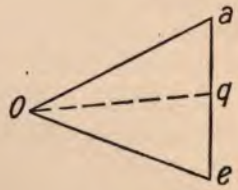

FIG. 10.

drawn. From any point in the line of action of F (Fig. 9) the two sides of the link-polygon lying in the spaces *a* and *e* are drawn parallel to O*a* and O*e* respectively, to cut the lines of action of F_1 and F_2 respectively. Thus the third side of the link-polygon lying in the space *q* is determined; and the radius vector O*q* (Fig. 10) being drawn parallel thereto, determines the magnitudes *qa* and *eq* of the forces F_1 and F_2 respectively.

Given a number of parallel forces acting on a body, to find the magnitudes of two parallel forces acting at two given

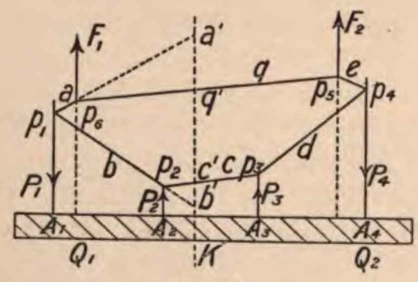

FIG. 11.

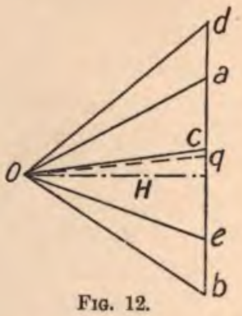

FIG. 12.

points, Q_1 and Q_2, which will produce equilibrium. The solution is effected by a combination of the constructions of

Figs. 7–10. The complete force- and link-polygons are each closed, and are clearly shown in Figs. 11 and 12.

Facility and accuracy of drawing force- and link-polygons for any given data are essential to a thorough knowledge of the subject. For this reason, a few examples are added at the end of this chapter.

(11) **Couple Diagram, or Bending-moment Diagram.**—A horizontal shaft subjected to various loads, P_1, P_2, . . . and reactions F_1 and F_2 at the bearings, is subjected to bending stresses. The straining action at any section of the shaft is the sum of the straining actions of all the forces on one side of that section.

In Fig. 11 the straining action at section K is the sum of the straining actions of the forces P_1, F_1, P_2, acting at A_1, Q_1, A_2. But the force P_1 at A_1 is equivalent to an equal force P_1 at K plus the couple of transference $P \times A_1K$; with similar expressions for the forces F_1 and P_2 acting at Q_1 and A_2 respectively. Thus the straining actions at K are a shearing force = algebraic sum of

$$P_1,\ F_1,\ P_2 = qc \text{ (Fig. 12)},$$

and a bending moment = algebraic sum of

$$P_1 \times A_1K, \qquad F_1 \times Q_1K, \qquad P_2 \times A_2K.$$

Let the sides a, b, c, and q of the link-polygon, produced if necessary, cut the vertical line through K at a', b', c' and q' respectively. Let H be the horizontal distance of O (Fig. 12) from the line of loads measured to the same scale as the loads. The triangles $a'p_1b'$ and aOb have their sides parallel, and are therefore similar figures.

Therefore $$\frac{ab}{H} = \frac{a'b'}{A_1K},$$

or $$\overline{ab} \times \overline{A_1K} = \overline{a'b'} \times H$$

But $\overline{ab} \times \overline{A_1K}$ is the bending moment on the section K due to the force P_1. Therefore the bending moment at K due to P_1 is $= a'b' \times H$.

By similar reasoning, the bending moment at K due to F_1 is

$q'a' \times H$; that due to P_2 is $b'c' \times H$. Finally, the bending moment at K is

$$H \times (q'a' + a'b' + b'c') = H \times q'c'.$$

Thus, the bending moment on any section of the shaft is equal to the intercept of the link-polygon by a vertical at that section, multiplied by H. By choosing a suitable scale, therefore, the vertical intercepts of the link-polygon may be regarded as a bending-moment diagram.

Bending Moment in Two Longitudinal Planes at Right Angles.—At any transverse section of a shaft the bending stress may be expressed as two component bending moments in two longitudinal planes at right angles. Each component bending moment is a vector quantity; the resultant bending moment, in magnitude and direction, is therefore determined by vector addition. If M_x and M_y are the component bending moments; the resultant is

$$M = \sqrt{M_x^2 + M_y^2}.$$

(12) **Scales of the Diagrams.**—If the linear scale be 1 inch $= x$ feet, the scale of the force-diagram 1 inch $= y$ lbs., and the polar distance is H inches or Hy lbs., then one inch ordinate of the couple or moment diagram represents a moment of x ft. $\times$ Hy lbs. That is, the scale of moments is one inch $= xy$H foot-lbs.

(13) **Mass-centre.**—The *centre of gravity* of a rigid body, or of a system of bodies of unchanging relative configuration, is the point always traversed by the resultant weight of the body or system of bodies. If the *weight* of a particle is proportional to its *mass*, the centre of gravity is also the *mass-centre*. This is sensibly the case for all problems relating to masses on the earth's surface, although it is not necessarily true for astronomical mechanics; weight, that is gravitational attraction between two masses, varying inversely as the square of the distance between them.

More generally and strictly, the *mass-centre* may be briefly defined as the *average position* of a number of particles all of equal mass, comprising the rigid body, or system of bodies. Taking any point, O, as reference, the position of each particle of mass m_1 is represented by a vector r_1, and

the position of the mass-centre, G, of a number of particles will be

$$\text{Vector OG} = \frac{\text{vector sum } (m_1r_1 + m_2r_2 + \ . \ . \ .)}{\text{sum } (m_1 + m_2 + \ . \ . \ .)}$$

If the reference point O is coincident with G, the vector OG is zero, and therefore

$$\text{Vector sum } (m_1r_1 + m_2r_2 + \ . \ . \ .) = 0.$$

(14) **Moment of Inertia.**—When a rigid body is capable of rotary motion about a fixed axis, we have already seen that a couple is required to cause it to turn. That is, in rotary motion, a *couple* is analogous to a *force* in linear motion; angular acceleration is analogous to linear acceleration in linear motion: what is analogous to *mass* in linear motion? The answer is, the *moment of inertia* of the body relative to the axis of rotation.

Consider a heavy particle of mass m to be constrained by a rigid frame of no mass, so as to turn about the axis O (Fig. 13), the distance of m from the axis being r. The particle will then move in a circular path of radius r. Let a constant force f act on the particle, always in the direction of the tangent, *i.e.* at right angles to the radius. The linear acceleration a of the particle is then determined by the equation $f = ma$. The angular acceleration ϕ of the rigid frame and particle is $\frac{a}{r}$; *i.e.* $a = \phi r$.

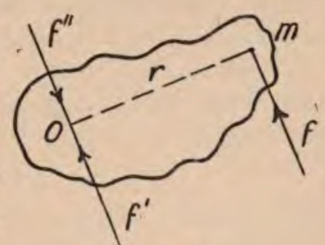

FIG. 13.

Draw at O, two equal but opposite forces f' and f'', each equal to f. These do not influence the rotary motion, since they directly neutralize each other. But f and f'' constitute a couple c, which turns the rigid body; the remaining force f' presses the rigid frame on the bearings or constraint at O.

But $$c = fr = mar = mr^2\phi.$$

Let $mr^2 = i$;

then $$c = i\phi, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

an equation analogous to

$$f = ma.$$

i is called the moment of inertia of the particle m, with respect to the axis O. Similarly, if there are a number of particles of masses $m_1, m_2, m_3 \ldots$ fixed in the massless rigid frame at radii $r_1, r_2, r_3 \ldots$ respectively from O, the couples $i_1, i_2, i_3 \ldots$ required to give them the common angular acceleration ϕ, are determined by equations similar to (1), the factor ϕ being common to all.

A heavy rigid body may be considered as made up of a number of such particles, and therefore the couple C required to give it the angular acceleration ϕ is

$$\begin{aligned} C &= (m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots)\phi \\ &= I\phi. \end{aligned}$$

Therefore, the moment of inertia I of the heavy rigid body, relative to the axis O, is defined by taking each element of mass, multiplying it by the square of its distance from the axis, and adding all the products. In the notation of the integral calculus, $I = \int r^2 dm$.

Since each element of I is a mass multiplied by the square of a length, I itself must have the same dimensions, *i.e.*

$$(I) = (m)(l)^2.$$

Or, the unit I is unit mass × square of unit distance, lb.-ft.^{2} in the British system of units. If M is the total mass of the rigid body, we may write $I = Mk^2$, where k is called the *radius of gyration.*

The student must be careful to notice that k^2 is not a vector quantity. If two equal particles, m_1 and m_2, are at the same radius, but in opposite directions from O, $r_1 = -r_2$, and the vector $m_1r_1 = -$ vector m_2r_2. But $r_1^2 = (-r_2)^2$, and $m_1r_1^2 = m_2r_2^2$. Thus, the radius of gyration, k, of a body is not associated with any particular direction; k^2 and I are *scalar* quantities, and are always positive.

The following table gives the values of I and k^2 for a few bodies of most frequent occurrence.

TABLE I.

MOMENTS OF INERTIA.

Body	Axis of Reference	I	k^2
Thin tube of circular section, radius r	Axis of tube	Mr^2	r^2
	On surface of tube, parallel to axis of tube	$2\ Mr^2$	$2r^2$
Circular disc of uniform thickness, radius r	Through centre, perpendicular to disc	$\frac{Mr^2}{2}$	$\frac{r^2}{2}$
	At circumference, perpendicular to disc	$\frac{3}{2}Mr^2$	$\frac{3}{2}r^2$
(Flywheel rim) Thick circular ring of rectangular section, outside and inside radii, r_1 and r_2	Axis of ring	$\frac{M(r_1^2+r_2^2)}{2}$	$\frac{(r_1^2+r_2^2)}{2}$
Rod of uniform section length l, transverse dimensions small in comparison with l	Through middle, perpendicular to axis	$\frac{Ml^2}{12}$	$\frac{l^2}{12}$
	Through end, perpendicular to axis	$\frac{Ml^2}{3}$	$\frac{l^2}{3}$

(15) **Comparison between Linear and Rotary Motion.**—From what has been already said, it should be clear to the student that the mathematics of rotary motion is precisely the same as that of linear motion; the three fundamental elements of the latter being time, length, and mass, the corresponding elements of rotary motion being time, angle, and moment of inertia. Any formula in linear motion has its analogue in rotary motion. Many engineering students who can easily solve problems in linear motion are hopelessly at sea with problems in rotary motion. The following table of analogies may be useful to them.

TABLE II.

Analogies between Linear and Rotary Motion.

Linear Motion: Name of Unit.	Linear Motion: Dimension of Unit.	Relation between linear and rotary motion.	Rotary Motion: Name of Unit.	Rotary Motion: Dimension of Unit.
Time, t	sec.	Same	Time, t	sec.
Space, s	foot	$\theta = s/r$	Angle, θ	. . 1 . .
Mass, m	lb.	$I = \Sigma mr^2$	Moment of inertia, I	lb.-ft.^{2}
Velocity, $v = \frac{s}{t}$	$\frac{\text{ft.}}{\text{sec.}}$	$\omega = \frac{v}{r}$	Angular velocity, $\omega = \theta/t$	$\frac{1}{\text{sec.}}$
Acceleration, $a = \frac{v}{t}$	$\frac{\text{ft.}}{\text{sec.}^2}$	$\phi = \frac{a}{r}$	Angular acceleration, $\theta = \omega/t$	$\frac{1}{\text{sec.}^2}$
Momentum, mv	$\frac{\text{lb.-ft.}}{\text{sec.}}$	$I\omega = \Sigma mvr$	Angular momentum, $I\omega$ (moment of momentum)	$\frac{\text{lb.-ft.}^2}{\text{sec.}}$
Force, $f = ma$	$\frac{\text{lb.-ft.}}{\text{sec.}^2}$	$C = fr$	Couple, $C = I\phi$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^2}$
Impulse, $ft = mv$	$\frac{\text{lb.-ft.}}{\text{sec.}}$	$Ct = \Sigma frt$	Angular impulse, $Ct = I\omega$ (moment of impulse)	$\frac{\text{lb.-ft.}^2}{\text{sec.}}$
Work done, $E = fs$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^2}$	Same	Work done, $E = C\theta$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^2}$
Kinetic energy, $E = \frac{mv^2}{2}$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^2}$	Same	Kinetic energy, $E = \frac{I\omega^2}{2}$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^2}$
Power, $\frac{E}{t} = fv$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^3}$	Same	Power, $\frac{E}{t} = C\omega$	$\frac{\text{lb.-ft.}^2}{\text{sec.}^3}$

Linear Motion.	Rotary Motion.
Uniform Acceleration.	
$v = v_0 + at$	$\omega = \omega_0 + \phi t$
$s = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\phi t^2$
$v^2 = v_0^2 + 2as$	$\omega^2 = \omega_0^2 + 2\phi\theta$
Centrifugal force, f	Gyroscopic couple, C
Impulse, ft	Angular impulse, Ct
Momentum, mv	Angular momentum, $I\omega$
Angle described, $\theta = ft/mv$	Angle of precession, $\theta = Ct/I\omega$
Angular velocity, $\omega = f/mv$	Speed of precession, $\Omega = C/I\omega$
$f = mv\omega = mv^2/r = m\omega^2 r$	$C = I\omega\Omega$

(16) **Angular Momentum.**—The linear momentum of a particle of mass m, moving with velocity v, is mv.

The angular momentum of a body rotating with angular velocity ω, and having moment of inertia I relative to the axis of rotation, is $\mathrm{I}\omega$. If the body is a heavy particle at radius r, $\mathrm{I} = mr^2$, and $\omega = \frac{v}{r}$; therefore its angular momentum is mvr. More generally, if m is the mass of a particle, r its perpendicular distance from any axis O, and v the component of its velocity in the direction mutually perpendicular to the axis and the perpendicular, its angular momentum (or *moment of momentum*) relative to that axis is mvr.

(17) **Plane Motion of a Rigid Body.**—The piston and connecting-rod of an engine have plane motion, *i.e.* each point moves in one plane at right angles to the axis of the crank-shaft. These bodies are also usually symmetrical about the plane of motion of their mass-centre. The following discussion is limited to the plane motion of such symmetrical bodies.

In Fig. 13, the force f', due to the particle m, presses the rigid frame on the axis O. $f' = ma = mr\dot{\phi}$, and is a vector quantity, its direction being at right angles to the radius r. Therefore, the total pressure on the axis due to the particles, $m_1, m_2, m_3, \ldots$ is

$$\text{Vector sum } (m_1r_1 + m_2r_2 + \ldots)\phi$$

If O is at the mass-centre of the body, this vector sum is zero, and there is no pressure on the axis.

If, however, the body is non-symmetrical about the plane of its motion, like a crank-shaft with two cranks at 180°, the resultant of all the pressures f' due to the particles comprising the body will be a couple. This will be discussed more fully.

Conversely, if a rigid body free to move is acted on by a couple, it will turn about an axis passing through its mass-centre.

Let Fig. 13 represent a heavy rigid body free to move with plane motion under the action of the external force f,

and let O be its mass-centre. Introduce as before the two forces f' and f'' at O, equal, opposite, and parallel to f. The force f' produces motion of the mass-centre, the couple f and f'' produce rotation about the axis O. These two motions take place independently and simultaneously.

Hence, the motion of the mass-centre of a rigid body under the application of any force is the same as if an equal parallel force were applied to the whole mass collected at the mass-centre. The motion of rotation of a body about an axis through its mass-centre, due to an external force, is the same as if the mass-centre were fixed.

The converse of these two theorems is true. Thus, the linear motion of the mass-centre of a connecting-rod, and the angular motion of the rod about an axis through its mass-centre, is determined at any instant by the constraint of the mechanism. The external force required to produce these motions is determined by the above theorems. This external force is evidently the resultant of the pressures on the crank-pin and cross-head ends of the rod, and is equivalent to an equal parallel force at the mass-centre of the rod, together with a couple of transference.

Plane Motion of a System of Bodies under no External Force.—Consider two bodies, A and B, of a system (Fig. 14) exerting mutual force at any instant. The force f_1 exerted by B on A, is equal, but opposite to the force f_2 exerted by A on B. Let t be the time of action, m_1 and m_2 the masses of the bodies, V_1 and V_2 the changes in the velocities of their mass-centres G_1 and G_2

Then $$f_1 t = m_1 V_1$$
$$f_2 t = m_2 V_2$$
also $$f_1 = -f_2$$
Therefore $$m_1 V_1 = -m_2 V_2, \text{ or } (m_1 V_1 + m_2 V_2) = 0.$$

That is, the total momentum is unaltered by the mutual action of the two bodies.

(18) **Conservation of Momentum.**—Generalizing from the above, *the resultant momentum of a system of bodies remains constant*, if no external force acts on the system.

This may also be expressed thus: The joint mass-centre of a system of bodies remains at rest or moves uniformly in a straight line, if no external forces act on the system; quite independently of the motions or connections of the bodies among themselves.

Conservation of Angular Momentum.—Let G be the common mass-centre of the two bodies (Fig. 14). Introduce two forces f_1' and f_2' at G, equal, opposite, and parallel to f_1 and f_2; f_1 and f_1' constitute a couple which in time t produce a change of angular momentum of the body A. The equal opposite couple formed by f_2 and f_2' produce an equal opposite change of angular momentum of the body B. Thus, the total angular momentum remains unchanged.

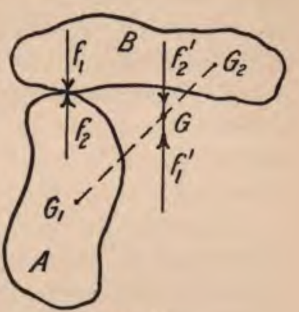

Fig. 14.

Generalizing, *the resultant angular momentum of a system of bodies about any axis remains constant*, if no external force or couple acts on the system.

The above results are the fundamental dynamical theorems for the discussion of the problem of engine balancing. Imagine an engine direct coupled to a dynamo on a common bedplate, which can be suspended by springs of such resilience that we may neglect the variation of load on them due to a slight extension. The external forces, the weight of the apparatus, and the reaction of the springs, are in equilibrium; the resultant external force is zero. Therefore, the joint mass-centre of the engine and dynamo is at *rest*. But the moving parts of the engine and dynamo may have their common mass-centre in motion. In this case the mass-centre of the remainder of the apparatus must move, and the bedplate will move cyclically during each revolution of the shaft. The engine is then said to be unbalanced, an unbalanced force existing at the bearings.

The common mass-centre of the moving parts may be at rest, but if their resultant angular momentum about any axis

is not always zero, that of the bedplate and fixed parts of the apparatus must also vary, and the bedplate will oscillate. Take, for example, an axis mutually at right angles to those of the cylinders and the crank-shaft. If the masses fixed to, and revolving with, the shaft are symmetrically disposed (balanced), their resultant angular momentum about this axis is zero. Suppose the engine to have two cylinders at distance l, and cranks at 180°. As one piston and connecting-rod moves towards the crank-shaft, the other moves away from it, the resultant angular momentum relative to the said axis is mvl, where mv is the resultant linear momentum of one piston and connecting-rod, and varies cyclically during each revolution. Therefore, the bedplate oscillates longitudinally about this axis.

If a counterbalance mass, m, is attached to the crank-shaft, opposite each crank-pin, the resultant angular momentum of the pistons, connecting-rods, and counterbalance masses about the same axis is approximately zero, since the linear momentum of each piston and rod is nearly equal and opposite to that of its counterbalance mass. But now, with reference to an axis parallel to the cylinders, the angular momentum of the pistons is always zero, since the component velocity at right angles to this axis is zero, the angular momentum of the two counterbalance masses about this axis varies cyclically during each revolution, and the bedplate, therefore, oscillates longitudinally about this axis.

Again, if the speed of the shaft is constant, its angular momentum, relative to its axis, is constant. But if the engine has only one cylinder and one connecting-rod, the angular momentum of the rod is successively zero, a maximum, zero, a minimum (or negative value), zero, during each revolution. The angular momentum of the remainder of the apparatus must, therefore, at each instant be equal but opposite to that of the connecting-rod. Therefore, the bedplate will oscillate about an axis through the mass-centre parallel to the axis of the shaft.

(19) **Kinetic Energy of a System of Bodies.**—Analogous to the expression $mv^2/2$, for the kinetic energy of a mass

m moving with linear speed v, the kinetic energy of a rotating body is $I\omega^2/2$, I being its moment of inertia about the axis of rotation, ω its angular speed. Using the gravitation unit of force, the *angular energy*, as it may be called in distinction to the *linear energy*, is $I\omega^2/2g$.

The total energy of a rigid body is the sum of the linear energy, due to the mass being concentrated at the mass-centre, and the angular energy due to its angular speed about an axis through its mass-centre.

Further, if V is the velocity of the mass-centre, V_x and V_y, the component velocities in two directions mutually at right angles, since $V^2 = V_x^2 + V_y^2$, the linear energy is the *sum* of the component energies in two directions mutually at right angles. Thus, the kinetic energy of the connecting-rod of an engine is expressed as the sum of three quantities.

The kinetic energy of a system of bodies may be expressed as the sum of (A) the linear energy of the total mass concentrated at the mass-centre of the system, and moving with the same linear speed; (B) the sum of the linear energies due to the linear speeds of the individual bodies relative to the mass-centre of the system; (C) the sum of the angular energies of the various bodies due to their angular speeds about axes through their own mass-centres. Thus, for example, if a motor-car weighing, with passengers, 2 tons, is moving at the speed of 30 miles an hour (44 ft. per second), the term A above is $\frac{2 \times 44^2}{2 \times 32 \cdot 2}$ foot-tons. The term B would include the linear energy of the pistons and connecting-rods relative to the engine frame. The term C would include the angular energies of the road-wheels, crank-shaft, and flywheel, etc., relative to their axes of rotation, also the energies of the angular swings of the connecting-rods relative to their own mass-centres.

Conservation of Energy.—*Energy cannot be created or destroyed;* it may be transformed from one kind to another, and it may be dissipated so as to be lost for all practical purposes. The source of energy which a heat-engine converts into mechanical work is in the chemical energy latent

in the fuel, and which by combustion in the furnace of the steam boiler, or in the cylinder of an internal-combustion engine, is converted into heat energy. The work done in the cylinders of an engine during any interval of time may be expended in (1) doing useful work in overcoming the resistance of the crank-shaft; (2) overcoming frictional resistances of the mechanism; (3) increasing the kinetic energy of the moving parts; (4) in producing elastic deformations of the parts considered as springs, *e.g.* twisting the shaft. If the work done in the cylinders during any time interval is greater than items (1) and (2) above, the excess energy is stored up as items (3) and (4); this energy being restored during a time interval when the work done in the cylinders is less than items (1) and (2).

(20) **Internal and External Forces.**—The whole system of forces in an engine, even with only one cylinder, is exceedingly complex. The relative motion of the parts is determined by the nature of the mechanism, the actual forces exerted between two parts, say, for example, a crank-pin and connecting-rod end may be due partly to inertia effects, partly to the pressure exerted by the working fluid on the piston. In the problem of engine balancing we are concerned with the resultant or couple force tending to move the engine frame. The complete engine with its moving parts, and fixed parts comprising cylinders, bed-plate, bearings, etc., together with the working fluid, form a system of bodies. We have seen above (sect. 18) that no mutual action between these bodies has any influence on the motion of the mass-centre of the complete engine, which therefore remains at rest; that is, the system of forces due to the presure of the steam, gas, or water, is a set of internal forces in equilibrium. If the steam presses downwards on the piston, this pressure is transmitted by piston-rod, connecting-rod, and crank to the bearing forming part of the frame. But, simultaneously, the steam presses upwards on the cylinder-head, which is also rigidly connected to the frame. These two forces on the frame being equal and opposite, balance each other, provided there is no force absorbed to

accelerate or retard the heavy moving parts. Considering now the moving parts of the engine, if the mass-centre moves, as the shaft rotates, the mass-centre of the *fixed* part of the engine must also move so as to keep the joint mass-centre at rest, and there is said to be an unbalanced force in the engine. If the bedplate is bolted down so as to prevent this actual motion, then the force exerted by the holding-down bolts varies during each revolution of the shaft; constituting a varying *external* force.

Thus we see that a detail study of mutual pressures of crank-shaft and main-bearings, crank-pin and big-end of connecting-rod, cross-head and slides, etc., however necessary to ensure satisfactory working of the engine, is not required for the study of balancing the engine.

Perhaps a simple analogy may make this fundamental principle more readily understood by the student. The wheel of a road vehicle has to transmit the load from the axle to the ground. It is acted on by (in the simplest case) two external forces, the downward pressure of the axle, and the upward reaction of the ground, which must, of course, be equal and opposite when the wheel has no vertical acceleration. The internal forces between the various parts comprising the wheel (tyre, felloe, rim, spokes, nave) may be exceedingly complex, but are in equilibrium among themselves, and have no direct influence on the external forces. The wheel may be loosely or tightly built; the spokes of a wood wheel are in compression, the rim in tension; conversely for a wire-spoke wheel.

Exercises.

1. Two weights each 20 lbs. are fastened to an iron bar at a distance apart of 3 feet between centres, forming a bar-bell. The bar-bell is suspended, and set rotating in a horizontal plane with a speed of 150 revolutions per minute. Find the tension on the bar.

Ans. 861 lbs.

2. Prove the formula giving the radial acceleration of a particle traversing a circle of radius R feet with a uniform speed of V feet per second, and thence find the speed of horizontal rotation of a heavy ball swinging by a string, when the string makes an angle of 30° with the vertical. (Inst. C.E., Feb. 1907.) *Ans.* $\omega = \sqrt{\dfrac{g}{\sqrt{3}R}}$

3. The connecting-rod of an engine is inclined at an angle of 30° to the crank. Compare the torque on the crank-shaft with the torque when the connecting-rod is at right angles to the crank; the pressure on the crank-pin being the same in both cases. *Ans.* 1 : 2.

4. A beam 20 feet long is supported on two supports 3 feet from each end of the beam. Weights of 10 lbs. and 20 lbs. are suspended from the two ends of the beam. Draw to scale, the bending-moment and shearing-force diagrams; and in particular estimate their values at the central section of the beam. (Inst. C.E., Feb. 1905.)
Ans. 2·14 lbs., 45 lb.-ft.

5. How much energy is stored up in a 3-foot thin rod weighing 4 lbs., and which is revolving at 140 revolutions per minute about an axis through its centre and perpendicular to its length? (Inst. C.E., Oct. 1900).
Ans. 10 foot-lbs.

6. Explain the meaning of the term "centrifugal force." With what speed must a locomotive be running on level railway lines forming a curve of 968 feet radius, if it produces a horizontal thrust on the outer rail equal to $\frac{1}{64}$ of its weight? (Inst. C.E., Feb. 1903.)
Ans. 15 mile/hour.

7. Estimate the super-elevation which ought to be given to the outer rail when a train moves round a curve of 2000 feet radius at a speed of 60 miles an hour, the gauge being 4 feet $8\frac{1}{2}$ inches. (Inst. C.E., Feb. 1904.) *Ans.* 6·8 ins.

8. If a torque of 0·245 ton-foot is acting on a shaft which makes 100 revolutions per minute, find the H.-P. transmitted. (Inst. C.E., Feb. 1899.) *Ans.* 10·45.

9. A flywheel weighing 10 tons whose radius of gyration is 5 feet, rests on bearings 10 inches in diameter. If the coefficient of friction of the axle and bearings is 0·006, find the constant moment which must act upon the wheel to get up a speed of 20 revolutions per minute in one minute. (Inst. C.E., Feb. 1906.) *Ans.* 3·92 ton-inch.

10. An experimental flywheel has an external diameter of 18 inches, the rim is 2 inches wide and 1 inch thick. The arms and boss may be neglected. A weight of 100 lbs. is attached to a cord wound round the axis. Find (1) the velocity of a point on the outside of the rim if the weight falls 15 feet freely under gravity, neglecting friction; (2) how many revolutions the wheel will make before being brought to rest by a braking force of 10 lbs. applied to the rim. Weight of cast iron is 450 lbs.-ft.^{3} (Inst. C.E., Feb. 1907.) *Ans.* M = 28 lb.,
$k^2 = 72·5$ in.^{2}, I = 14·1 lb.-ft.^{2}, $v = 27·8$ ft./sec., 26·2 revs.

11. An engine is running at 240 revolutions per minute when the steam is cut off, and the load removed at the same instant. The engine runs 300 revolutions before coming to rest. The flywheel weighs 2000 lbs. and has a radius of gyration of 3 feet. Find the moment of resistance—

assumed independent of speed; and if the engine indicate 14 H.-P. under normal working conditions, find the mechanical efficiency of the engines—the moment of resistance being assumed independent of load also. (Inst. C.E., Feb. 1905.) *Ans.* 94 lb.-ft., 0·694.

12. A thin circular disk, 12 inches radius, has a projecting axle $\frac{1}{2}$ inch diameter on either side. The ends of this axle rest on two parallel inclined straight edges inclined at a slope of 1 in 40, the lower part of the disc hanging between the two. The disc rolls from rest, through 1 foot in $53\frac{1}{2}$ seconds. Neglecting the weight of the axle and frictional resistances, find the value of g. (Inst. C.E., Feb. 1905.)

Ans. 31·58 ft./sec.^{2}

13. Find an expression for the radius of gyration of a circular disc. An engine which develops 150 H.-P. on the shaft at its normal speed of 75 revolutions per minute (and whose card-area does not alter with the speed), has a disc flywheel 10 feet diameter, weighing 12 tons. How long will it take to get up speed from rest, there being no external load? (Inst. C.E., Oct. 1904.)

Ans. $k^2 = \frac{r^2}{2} = \frac{d^2}{8}$, 7·3 seconds.

14. A cast-iron flywheel 10 feet in diameter, with a rim 6 inches by 6 inches, is rotating freely on a shaft at the rate of 100 revolutions per minute. A brake which exerts a frictional retardation of 100 lbs. is applied to its rim for 20 seconds. Find how much the speed of the flywheel is reduced (neglecting the weight of its arms). (Inst. C.E., Feb. 1900.) *Ans.* 40 revs./min.

15. Show how the resultant of two couples can be found, if they act in (i.) parallel planes, (ii.) planes not parallel. If four couples of 10, 18, 25, and 10 foot-lb. act on a rigid body in planes which all intersect in a line, but are inclined at angles of 30°, 60°, 90° respectively with the plane of the first-mentioned couple, find their resultant. (Inst. C.E., Feb. 1900.)

Ans. 80·5 ft.-lbs.; in plane inclined 62·2°.

CHAPTER III

INERTIA FORCES OF REVOLVING MASSES

In this chapter are discussed the inertia forces due to different masses fixed to and revolving with a shaft, the angular speed being uniform. Each inertia force is represented by a localized vector drawn from the centre of the mass perpendicular to the axis of the shaft. The various vectors, therefore, revolve with the shaft, but preserve the same relative position. Therefore, they may be compounded and resolved as if the shaft were at rest. The various components or resultants being fixed relative to the shaft, they, of course, revolve with the shaft; each vector turning in a *transverse* plane.

(1) **Mass-moment.**—In sect. 1, Chap. II., it has been proved that a mass m, fixed at a radius r from the axis of a shaft turning with uniform angular speed ω, gives rise to a radial or centrifugal force of magnitude $mr\omega^2/g$. Similar expressions $m_2r_2\omega^2/g$, $m_3r_3\omega^2/g$, . . . hold for the inertia forces of masses m_2, m_3, . . . fixed to the same shaft at radii, r_2, r_3 . . . respectively. Each inertia force is represented by a localized vector drawn radially outwards from the axis of the shaft, passing through the centre of the mass. As the shaft rotates the various force vectors rotate with it. Since the factor ω^2/g is common to all the expressions for the inertia forces due to masses fixed to the same shaft, leaving the expressions m_1r_1, m_2r_2, m_3r_3 . . . for the individual masses, it will be convenient in discussing problems relating to inertia forces to deal with these latter expressions, instead of the actual inertia

forces, which, of course, vary as the speed ω of the shaft varies. m_1r_1, m_2r_2, m_3r_3 . . . are evidently localized vector quantities, and can therefore be compounded and resolved by the methods of Chap. II. m_1r_1 may be called the mass-moment of the mass m_1 at radius r_1. The vector diagrams, similar to Figs. 6, 8, 10, and 12, Chap. II., when drawn for mass-moments, may be called mr vector-polygons.

If the radii r_1, r_2 are unequal, the masses m_1, m_2 . . . may be replaced by corresponding masses M_1, M_2 . . . all at the same distance R from the centre of the shaft, provided $M_1R = m_1r_1$, etc. In complex problems, it will often be convenient to speak of the masses M_1, M_2 . . . etc., all at the same distance R, instead of the mass-moments m_1r_1, m_2r_2 . . .

(2) **Masses in the Same Plane of Revolution.**—The

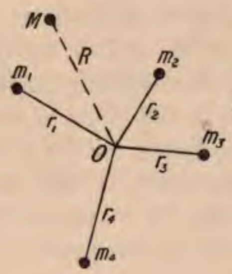

FIG. 1.

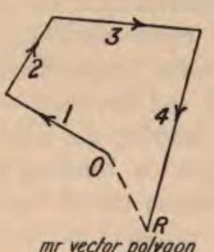

FIG. 2.

given masses being rigidly fixed to the shaft, in the relative positions shown in Fig. 1, the various values of mr are calculated, and the mr vector-polygon (Fig. 2) drawn, with the sides of length m_1r_1, m_2r_2 . . . drawn parallel to r_1, r_2 . . . respectively (Fig. 1). If O and R are the initial and final points of the mr vector-polygon, the resultant Σmr is determined by the vector $\widehat{OR}$. The inertia forces due to the given masses can be balanced by a single mass M at radius R (Fig. 1), the radius R (Fig. 1) being drawn parallel to the closing side $\widehat{RO}$ (Fig. 1) of the mr vector-polygon;

$$\text{Vector MR} = - \text{ vector } \Sigma mr.$$

As a particular example of the above, evidently a single mass m_1 at radius r_1 can be balanced by a mass m_2 at a radius r_2 opposite to r_1 (*i.e.* the angle between r_1 and r_2 is 180°); the mass-moments m_1r_1 and m_2r_2 being equal.

(3) ***mrl* Link-polygon.**—When the masses rigidly fixed to a rotating shaft, or the inertia forces due to them, are in different planes of revolution, they give rise to longitudinal couples, or bending moments, on the shaft. If the inertia forces are all in the same longitudinal plane, the methods discussed in sects. 10 and 11, Chap. II., can be employed. The inertia force $m_1r_1\omega^2/g$ due to a mass m_1 at radius r_1, in the plane of revolution A_1 (Fig. 11, Chap. II.), is equivalent to an equal parallel force in the plane of revolution K, plus the couple of transference $m_1r_1l_1\omega^2/g$, l_1 being the distance between the planes of revolution A_1 and K. Otherwise stated, the inertia force $m_1r_1\omega^2/g$ at A_1 produces on the section K an equal shearing force, and a bending moment $m_1r_1l_1\omega^2/g$ in the longitudinal plane passing through the axis of the shaft and the mass m_1 in the plane of revolution A_1. Instead of drawing the link-polygon for the actual longitudinal couples or bending moments, it will be convenient (for the reasons stated in sect. 1), to regard it as giving the various values of *mrl* at the various sections of the shaft. The link-polygon (Fig. 11, Chap. II.) may then be called the *mrl* link-polygon. The actual value of the longitudinal couple or bending moment at any section of the shaft is then obtained by multiplying the intercept of the *mrl* link-polygon at that section by the factor ω^2/g.

(4) **Masses in the Same Longitudinal Plane.**—The linear diagram (Fig. 3) shows the position of the given masses, m_1, m_2, m_3, m_4, all rigidly fixed to the shaft in the same longitudinal plane (the plane of the paper) at radii r_1, r_2, r_3, r_4, respectively, the distances between their planes of revolution being also given. Using the notation described in sect. 10, Chap. II., the *mr* vector-polygon is drawn, and the pole O taken at any convenient distance H from the line of the *mr* vectors. The *mrl* link-polygon is drawn below the linear

diagram, as explained in sect. 10, Chap. II. The resultant Σmr is measured by $\widehat{ae}$ in the mr diagram, and its plane is vertically over f, the point of intersection of the two sides of the mrl link-polygon respectively parallel to the initial and final radius vectors Oa and Oe of the mr diagram. The

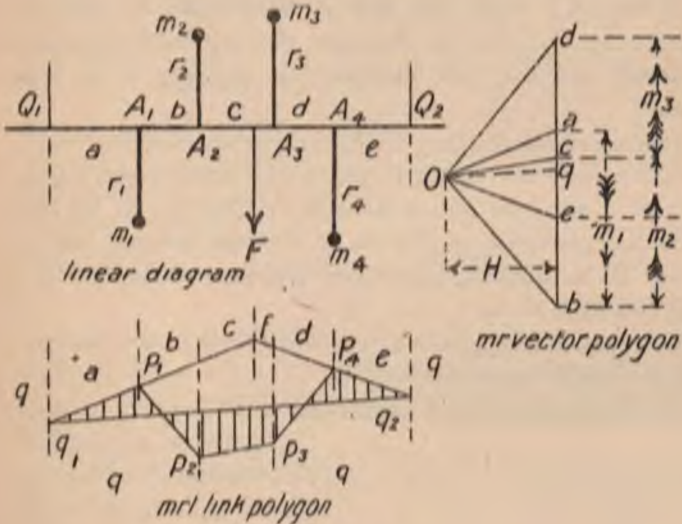

FIG. 8.

resultant inertia force F is equal to $\widehat{ae} \times \omega^2/g$, and is in the transverse plane of f. If the mr vector-polygon is closed, there is no resultant inertia force, but there will be a resultant inertia couple unless the mrl link-polygon is also closed.

(5) **Pressure on Bearings.**—If a single bearing were placed at the plane of f, the force of reaction on the shaft would be equal and opposite to F, and would equilibrate the inertia forces due to the given masses. If two bearings were placed at Q_1 and Q_2 respectively, the mrl link-polygon is finished as shown by the vertical lines, by drawing p_1q_1, and p_4q_2 parallel to Oa and Oe respectively, to meet the verticals through Q_1 and Q_2 at q_1 and q_2 respectively. The closing

line q_1q_2 of the link-polygon determines the radius vector Oq. Thus the reactions f_1 and f_2 of the bearings at Q_1 and Q_2 are $\widehat{qa} \times \omega^2/g$ and $\widehat{eq} \times \omega^2/g$ respectively.

(6) **Balancing of Masses in the Same Longitudinal Plane.**—Evidently, from Fig. 3, the given masses can be balanced by a *single mass* M placed at radius R in the plane of f such that MR is represented on the mr diagram by $\widehat{ea}$. For in this case the mr vector-polygon is closed, and the mrl link-polygon $p_1p_2p_3p_4\ f$ is also closed.

Instead of balancing by a single mass, as above described, it is often more convenient to use two balance masses, which may be in any two planes selected arbitrarily. If Q_1 and Q_2 be the planes of revolution of the two balance masses M_1 and M_2 at radii R_1, R_2 respectively, then evidently $M_1R_1 = \widehat{qa}$ and $M_2R_2 = \widehat{eq}$.

This method is applicable even when there is no resultant force, but only a resultant couple.

(7) **Example, Single Revolving Mass.**—Fig. 4 shows the graphical solution of this example, the bearings Q_1 and Q_2 being shown at unequal distances a and b from the plane of revolution of the mass m_1. If no balance masses are used, the inertia force due to m_1 at radius r_1 is $F = m_1r_1\omega^2/g$. The reactions f_1 and f_2 at the bearings Q_1 and Q_2 are evidently equal to $\frac{b}{(a+b)}F$ and $\frac{a}{a+b}F$ respectively.

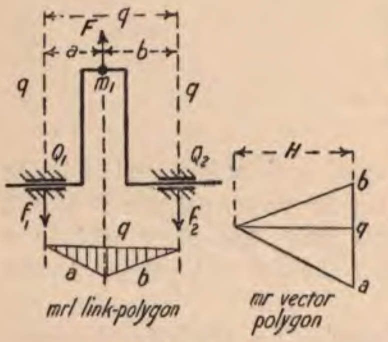

Fig. 4.

The single mass m_1 may be balanced by masses M_1 and M_2 at radii R_1 and R_2 in the planes of revolution Q_1 and Q_2 respectively

$$M_1R_1 = \frac{bm_1r_1}{(a+b)} \qquad M_2R_2 = \frac{am_1r_1}{(a+b)}$$

This case occurs in many practical problems of design, the mass of a crank being usually balanced by two masses, one on each side of the plane of revolution.

Fig. 5 shows the *mr* vector-polygon and *mrl* link-

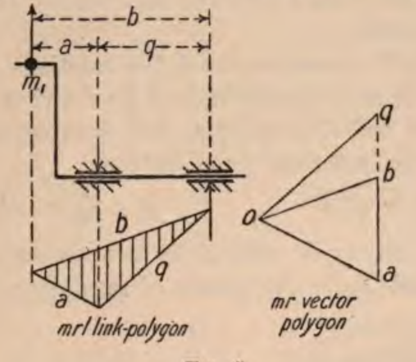

FIG. 5.

polygon when the mass is overhung, *i.e.* when the planes Q_1 and Q_2 are on the same side of the plane of revolution of m_1.

(8) **Two Cranks at 180°.**—Let the two masses m_1 and

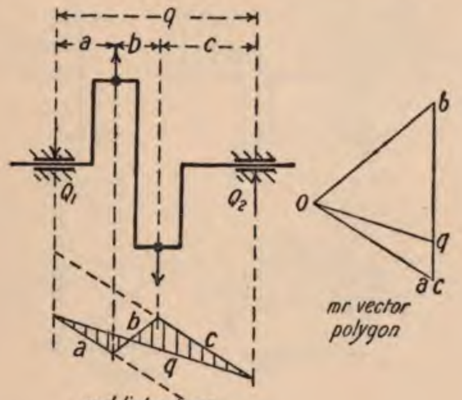

FIG. 6.

m_2 be equal, and be at equal radii, *i.e.* $m_1r_1 = m_2r_2$ (Fig. 6). The *mr* vector-polygon and *mrl* link-polygon are drawn as

shown in Fig. 6. The *mr* vector-polygon *abc* is closed; therefore there is no resultant inertia force. The *mrl* link-polygon *abc* is not closed, the two sides *a* and *c* being parallel; there is, therefore, a resultant couple, its *mrl* being measured by the vertical intercept between the lines *a* and *c* of the *mrl* diagram.

If no balance masses are used, the reactions f_1 and f_2 at the bearings Q_1 and Q_2 are obtained by drawing the closing line *q* of the *mrl* link-polygon, and drawing the parallel radius vector O*q* in the *mr* vector-polygon

$$f_1 = \widehat{qa} \times \omega^2/g \qquad\qquad f_2 = \widehat{cq} \times \omega^2/g$$

The balance may be effected by two masses M_1 and M_2 at radii R_1 and R_2 in the planes of revolution Q_1 and Q_2 respectively

$$M_1R_1 = \widehat{qa} \qquad\qquad M_2R_2 = \widehat{cq}$$

The shaded portion of the *mrl* diagram is the bending-moment diagram of the crank-shaft.

(9) **Four Cranks in the Same Longitudinal Plane.**—In a motor-car four-cylinder engine, the four masses revolving

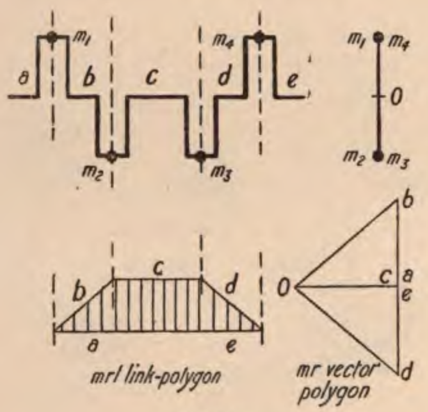

Fig. 7.

with the crank-shaft are equal and are at equal radii. The cranks are arranged as shown in Fig. 7, the distances *b* and *d* between pairs of centres being equal. In this case the

mr vector-polygon and *mrl* link-polygon are each closed, and there is, therefore, no unbalanced force or couple.

(10) **Masses in Different Longitudinal and Transverse Planes.**—In Fig. 8, the longitudinal elevation shows the

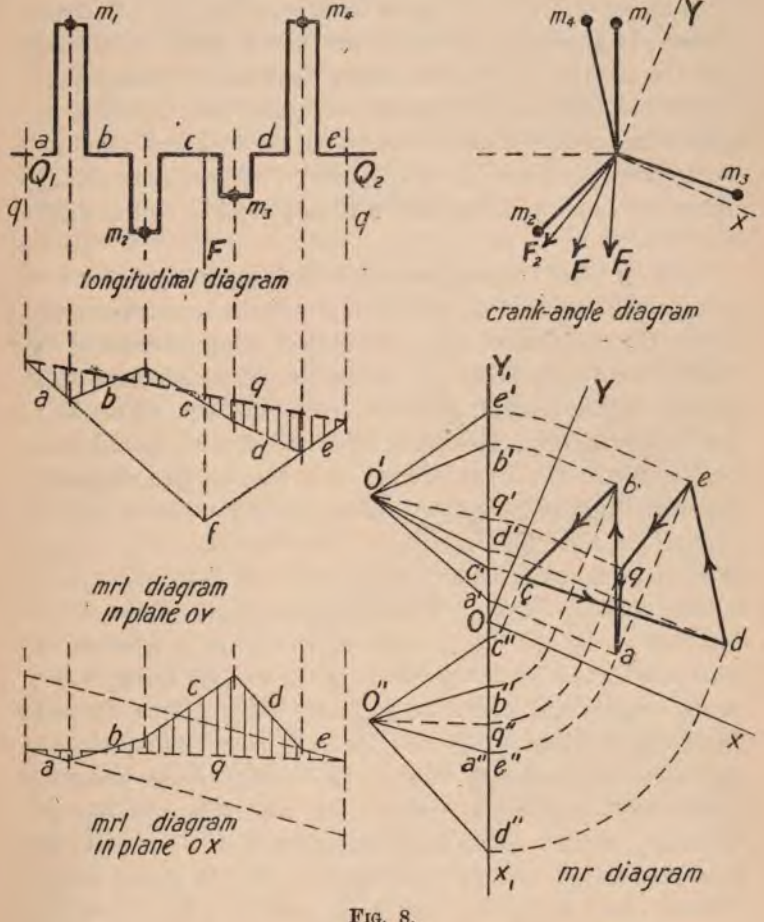

FIG. 8.

distances between the planes of revolution (transverse planes) of the given masses m_1, m_2, m_3, m_4, and the end elevation shows the radii r_1, r_2, r_3, r_4, and the relative angular position of the given masses. The end elevation may be called the

crank-angle diagram. If the inertia forces of the given masses do not balance, *i.e.* if they have a resultant force, or a resultant couple, or both, pressure at the bearings will be induced. If the frame supporting the bearings is bolted to a rigid foundation, the axis of the shaft cannot move; therefore, the external forces acting on the shaft must be in equilibrium. But the reactions of the bearings due to the inertia forces, combined with the inertia forces of the given masses, form the system of external forces acting on the shaft, and are, therefore, in equilibrium among themselves. The bearings are assumed at any two positions Q_1 and Q_2, and their reactions being determined, the straining action at any section of the shaft is determinate.

(11) ***mr* Vector-polygon.**—The various inertia forces for the given masses may all be transferred to the same transverse plane; the position of the chosen plane being immaterial as regards the forces, although the couples of transference will depend on the chosen position (sect. 6, Chap. II.). The inertia forces may, therefore, be considered all transferred to the plane of the paper on the crank-angle diagram. The magnitude and angular position of the resultant inertia force F is then determined by drawing the *mr* vector-polygon; the resultant *mr* being as before the closing line of the vector-polygon. The resultant inertia force F is equilibrated by the components f_1' and f_2' of the forces of reactions F_1 and F_2 at the bearings Q_1 and Q_2 lying in the same longitudinal plane at F. But the reactions F_1 and F_2 of the bearings at Q_1 and Q_2 may have two equal and opposite components f_1'' and f_2'' in a longitudinal plane at right angles to the plane of F, for then the resultant of f_1'' and f_2'' is zero. That is, f_1'' and f_2'' may be two equal and parallel but opposite forces, forming a couple in the longitudinal plane at right angles to F, or having its axis in the direction of F. Thus, in general, the inertia forces of the given masses have a resultant force and a resultant couple.

(12) ***mrl* Link-polygons. Resultant Couple.**—To determine the resultant couple, the procedure is as follows: Having drawn the *mr* vector-polygon, draw two axes, OX and

OY respectively at right angles and parallel to $\widehat{ae}$ (Fig. 8), the resultant *mr* vector. OX and OY may be considered as traces of longitudinal planes rotating with the shaft. Project the corners of the *mr* vector-polygon on these axes. Each *mr* may be resolved into two components in these planes. Next rotate the axis OY about O into the position OY_1, at right angles to the axis of the longitudinal diagram, carrying with it the component *mr* vector-polygon $a'b'c'd'e'$. Similarly rotate OX with the component *mr* vector-polygon $a''b''c''d''e''$. Taking any pole O′, and joining it to the corners $a'b'c'd'e'$ of the *mr* vector-polygon, the *mrl* link-polygon in the plane OY is drawn. The intersection of the sides a and e of the *mrl* link-polygon determines the position of the force F. The radius vector $O'q'$ parallel to the closing side q of the *mrl* link-polygon determines $\widehat{q'a'}$, and $\widehat{e'q'}$ which, multiplied by ω^2/g, give the components f_1' and f_2' of the forces at the bearings Q_1 and Q_2 in the plane of F.

The *mrl* link-polygon in the plane OX is then drawn in a similar manner. The two sides a and e are parallel—that is, they meet at infinity, and the resultant couple is measured by the vertical intercept between them. The equilibrating couple is formed by the forces f_1'' and f_2'' at the bearings Q_1 and Q_2; these are determined as explained above, by the closing side q of the *mrl* link-polygon, and are respectively $q''a''$ and $e''q''$ multiplied by ω^2/g.

If the *mr* vector-polygon is closed, the axes OX and OY for the *mrl* diagrams may be taken in any position; it is most convenient then to take them respectively parallel and at right angles to the axis of the longitudinal diagram. The points a and e of the *mr* vector-polygon, in this case, coincide; the constructions for finding the point q in the *mr* vector-polygon are exactly the same as shown in Fig. 8; but F_1 and F_2 are equal, parallel, and opposite.

(13) **Pressure on Bearings.**—Lastly, projecting and rotating q' and q'' back into the *mr* vector-polygon, we obtain the point q; and the vectors $\widehat{qa}$ and $\widehat{eq}$ multiplied by ω^2/g give the actual magnitudes and directions of the forces F and F_2 at the bearings Q_1 and Q_2 respectively.

At any section K the vertical intercepts on the link-polygons give the component bending moments at that section. If desired, the resultant bending moment at the section may be found in magnitude and direction by compounding as in sect. 11, Chap. II.

(14) ***mrl* Vector-polygon. Resultant Transference Couple.**—Reverting to Fig. 8, the inertia forces due to the given masses are equivalent to equal parallel forces all lying in the same transverse plane, together with the various transference couples. Each transference couple lies in the axial plane of the mass, and the axis of the couple is at right angles to the axial plane of the mass. The moment of the transference couple of the mass m_1 is $m_1r_1l_1\omega^2/g$, where l_1 is the axial distance between the plane of revolution of m_1 and the plane to which the inertia force is transferred. The transference couple, or the quantity $m_1r_1l_1$, can therefore be represented by a vector in a direction at right angles to the crank m_1. The resultant transference couple is therefore found by drawing an mrl vector-polygon, the sides of which are at right angles to the mr vector-polygon.

In problems relating to inertia forces of masses revolving with a shaft, the planes of the various transference couples all intersect in the same straight line, the axis of the shaft. The axes of the transference couples may therefore all be taken in the same plane at right angles to the axis of the shaft, that is, in a plane of revolution which may be the plane of transference. In actually drawing the couple vectors, it is more convenient to rotate the mrl vector-polygon through a right angle, so that its sides are then parallel to those of the mr vector-polygon. In drawing the mrl vector-polygon, attention must be paid to the algebraic sign of l, the distance of the mass from the chosen plane of transference. If the masses m_2 and m_3 are on opposite sides of the plane of transference, and vector $m_2r_2l_2$ is drawn in the same sense as vector m_2r_2, the vector $m_3r_3l_3$ must be drawn in the sense opposite to vector m_3r_3.

Fig. 9 shows the mrl vector-polygon for the masses

m_1, m_2, m_3, m_4, the masses M_1 and M_2 at Q_1 and Q_2 respectively (Fig. 8), the plane of transference being the plane of revolution of F. The vector $\hat{ae}$ is the resultant *mrl* for the four masses m_1, m_2, m_3, m_4, and therefore agrees with the intercept between the lines a and e of the *mrl* link-polygon in the plane OX (Fig. 8). The *mrl* vector $\hat{ae}$ is at right angles to the *mr* vector $\hat{ae}$. If at any plane of transference the couple is zero, the *mrl* polygon must be closed. It follows, therefore, that if the given masses are balanced among themselves, the *mrl* vector-polygon is closed, whatever plane of transference be chosen.

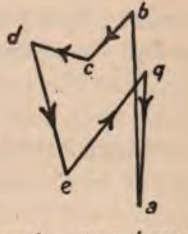

FIG. 9.

If at a certain plane of transference the *mrl* vector-polygon is closed, it does not necessarily follow that the given masses are balanced among themselves. If there is a resultant force, but no resultant couple, the *mrl* polygon will be closed when the plane of transference coincides with the plane of revolution of the resultant force, but not otherwise.

If there is no resultant force, but a resultant couple, the unclosed side of the *mrl* vector-polygon is the same, whatever plane of transference be taken.

(15) **Balancing of Revolving Masses.**—From the above discussion, it is evident that any number of masses rigidly fixed to and revolving with a shaft may, in general, be balanced by two masses in any two planes chosen at random. Thus, at the planes Q_1 and Q_2 (Fig. 8), if masses M_1 and M_2 be fixed in such positions that M_1R_1 and M_2R_2 are represented in the *mr* diagram by the vectors $\hat{qa}$ and $\hat{eq}$ respectively, the *mr* vector-polygon for all the masses, and both the *mrl* link-polygons are closed; the resultant force and resultant couple are each zero. There is then no inertia forces on the bearings.

(16) **Recapitulation of General Graphical Method.**—The methods discussed in sects. 10–15 above, and illustrated in Fig. 8, may be recapitulated thus—

Given any number of masses fixed to, and revolving with, a shaft. The inertia forces are equilibrated by the reactions of the bearings of the shaft. Two bearings at any assumed positions Q_1 and Q_2 evidently provide the necessary constraint. The problem graphically solved in Fig. 8 is therefore to find the magnitude and direction of the two equilibrating forces at the two positions Q_1 and Q_2. These equilibrating forces at Q_1 and Q_2 may be due either to (1) reaction of the bearings assumed at Q_1 and Q_2, or to (2) counterbalance masses placed in the planes of revolution at Q_1 and Q_2. Thus the problems of (1) finding the reactions at the two bearings, and (2) finding two counterbalance masses one in each of two given planes, are identical.

The resultant of the inertia forces due to the given masses reduces, in general, to a force and a couple, the axis of the couple being in the same direction as the force. The longitudinal and crank-angle diagrams defining the positions of the given masses being drawn, the procedure is as follows :—

1. Draw the *mr* vector-polygon, its sides being parallel to the corresponding cranks on the crank-angle diagram. The closing side $e\hat{a}$ of the *mr* vector-polygon is the *mr* for the equilibrating forces.

2. Draw an axis, OY, parallel to the closing side of the *mr* polygon, and an axis, OX, at right angles. Project the sides of the *mr* polygon on to OY and OX respectively. Swing the axes OX and OY with their various points, into directions OX_1 and OY_1 at right angles to the axis of the shaft in the longitudinal diagram.

3. Take a pole, O, in any convenient position, join it to the various points in OY_1, and draw the *mrl* link-polygon in plane OY. The intersection f of the first and last sides, a and e, of the link-polygon determines the plane of the resultant inertia force F due to the given masses. The intersections of the first and last sides, a and e, with the planes Q_1 and Q_2 respectively, determine the closing side q of the *mrl* link-polygon. Draw Oq' in the *mr* diagram parallel to the closing side q of the *mrl* link-polygon, thus determining the components in the plane OY of the *mr*'s at Q_1 and Q_2.

4. Similarly, draw the *mrl* link-polygon in the plane OX. The first and last sides, *a* and *e*, are parallel, the vertical intercept between them giving the *mrl* of the resultant couple due to the given masses. The intersections of *a* and *e* with Q_1 and Q_2 determines the closing side *q* of the *mrl* link-polygon. Draw Oq'' in the *mr* diagram parallel to *q*, thus determining the components in the plane OX of the *mr*'s at Q_1 and Q_2.

5. From q' and q'', by rotation and projection back into the *mr* vector-polygon, the point *q* is determined. $\widehat{qa}$ is the vector M_1R_1 at Q_1, $\widehat{eq}$ the vector M_2R_2 at Q_2.

6. At any transverse section, the vertical intercepts of the closed *mrl* vector-polygons, multiplied by ω^2/g give the component bending moments due to the inertia forces, in two planes at right angles. The resultant may be found by the method of sect. 11, Chapter II.

(17) **Example : Balancing a Locomotive Crank-axle.** —*Data:* Stroke 26″, distance between centres of cylinders (inside) 23″, distance between centres of driving-wheels, 5 ft., two cranks at right angles, mass considered fixed at centre of each crank-pin 240 lbs. Find positions and magnitudes of balance masses in driving-wheels.

Fig. 10 shows the complete graphical solution. The two cranks being drawn at 45° on each side of the vertical in the crank-angle diagram, the resultant *mr* for the two given masses is parallel to the axis OY in the *mr* diagram.

The values of the vectors $\widehat{qa}$ and $\widehat{eq}$ for M_1R_1 and M_2R_2 are independent of the scales to which the diagrams are drawn. They each measure 196 lb.-ft., *i.e.* a mass of 98 lbs. at a radius 2 ft. is required in each driving-wheel. The angle m_2OM_2 between the crank-pin and the balance mass in the corresponding driving-wheel measures 156°. But if it be desired to use the *mrl* diagrams to find the bending moment at any section of the shaft, it is convenient to use lbs. and ft. as the units. The scales used are shown at the foot of the diagram. In the original drawing from which Fig. 10 has been reduced, the scales were as follows : Linear

scale, 1 inch = 2 feet; *mr* scale, 1 inch = 200 lb.-ft.; H the polar distance was taken, 1 inch = 200 lb.-ft.; therefore the *mrl* scale is 1 inch = 400 lb.-ft^2.

The maximum ordinates of the *mrl* diagrams in the

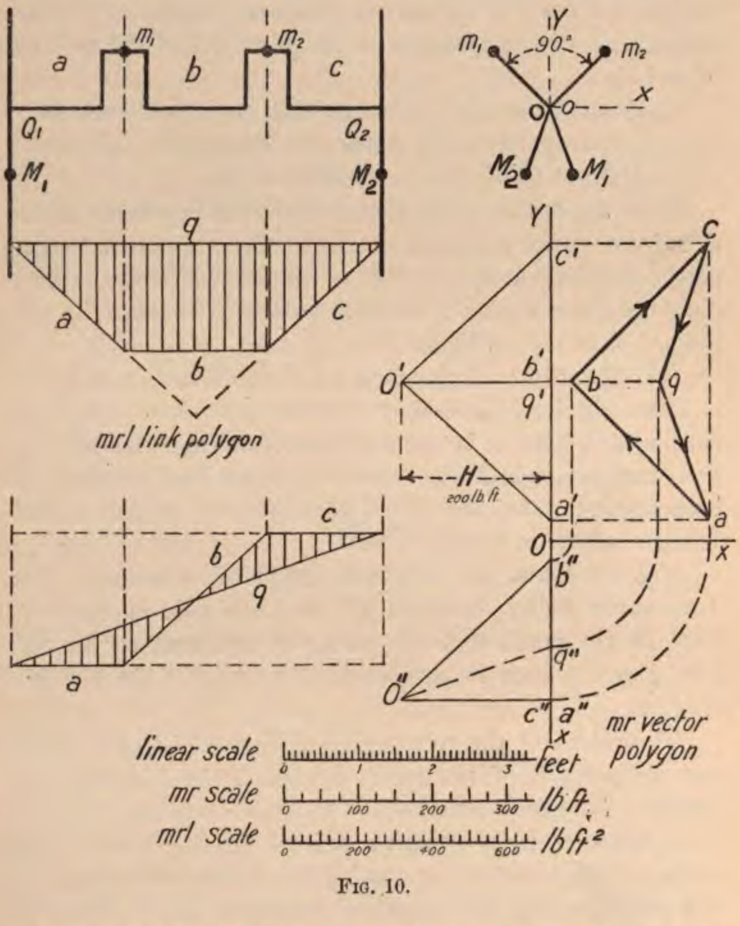

Fig. 10.

planes of OX and OY are at the crank-pin centres, and measure 108 and 288 lb.-ft.^{2} respectively. Combining them, the resultant *mrl* is

$$\sqrt{108^2 + 288^2} = 307 \text{ lb.-ft.}^2$$

If the driving-wheels are 6 ft. 6 in. diameter, *i.e.* 3·25 ft. radius, and the maximum speed is 80 miles an hour,

$$\omega = \frac{80 \times 5280}{3600 \times 3\cdot25} = 36\cdot1 \text{ radians per sec.}$$

$$\frac{\omega^2}{g} = \frac{36\cdot1 \times 36\cdot1}{32\cdot2} = 40\cdot7 \text{ ft}^{-1}.$$

The bending moment at the crank-pin due to the inertia forces is therefore

$$40\cdot7 \times 307 = 12{,}500 \text{ lb.-ft.}$$
$$= 150{,}000 \text{ lb.-in.}$$

In the above case of a *symmetrical* two-crank shaft, the position of the balance masses M_1 and M_2, relative to the given mases m_1 and m_2, can be quickly calculated. Let $M_1 = M_2 = m_1 = m_2$. Then, in the crank-angle diagram, the lines joining M_1 to M_2 and m_1 to m_2 must be parallel, and at equal distances on opposite sides of O. Also, the distances m_1m_2 and M_1M_2 are inversely proportional to the corresponding given axial distances between the planes of revolution. Thus the angular positions of M_1 and M_2 are determined; these remain unaltered, even if M_1 and M_2 are not equal to m_1 and m_2.

If the axle is left unbalanced, the unbalanced forces between the driving-wheels and rails will be due to the *mr* vectors $\widehat{aq}$ and $\widehat{qc}$ respectively. Generally, let m be equal to each of the unbalanced masses m_1 and m_2, r the radius of the cranks, l the distance between the planes of revolution of the cranks 1 and 2, L the distance between the planes of contact of the driving-wheels Q_1 and Q_2 with the rails.

The unbalanced *mr* vector $\widehat{ac}$ is in the plane OY, and the plane of rotation is midway between the two cranks. Its value is

$$\sqrt{2}mr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

As the axle rotates, this causes a variation of the total pressure of the two driving-wheels on the rails, at one instant diminishing the total pressure of adhesion, and 180°

later increasing the total pressure. Also, when the rotating vector $\widehat{ac}$ is horizontal, the unbalanced force is parallel to the rail, and is added to, or subtracted from, the horizontal driving effort due to the engine torque. This is equivalent to two equal parallel vectors rotating in the planes Q_1 and Q_2 respectively, each equal to

$$\frac{1}{\sqrt{2}}mr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The unbalanced *mrl* vector is H × the intercept y between the parallel lines a and c of the *mrl* link-polygon in the plane OX. From similar triangles

$$\frac{y}{l} = \frac{a''b''}{o''a''} = \frac{\frac{1}{\sqrt{2}}mr}{\mathrm{H}}$$

or

$$\mathrm{H}y = \frac{1}{\sqrt{2}}mrl.$$

that is, the unbalanced *mrl* vector is

$$\frac{1}{\sqrt{2}}mrl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When the plane OX rotating with the axle is horizontal, a swaying couple is produced; when OX is vertical, a transverse rocking couple. This *mrl* vector is equivalent to two equal opposite parallel *mr* vectors in the planes Q_1 and Q_2 respectively, each equal to

$$\frac{1}{\sqrt{2}}mr\frac{l}{\mathrm{L}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The *mr* vectors (2) and (4) in the plane of revolution Q_1 are at right angles to each other. Consequently, their resultant, which is $\widehat{dq}$ in the *mr* vector-polygon (Fig. 10), is of magnitude

$$mr\sqrt{\tfrac{1}{2}\left\{1 + \left(\frac{l}{\mathrm{L}}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and is the *mr* vector which determines the variation of load on each rail, due to the unbalanced revolving masses.

If the expressions (1), (3), and (5) be multiplied by ω^2/g, they give respectively the greatest variation of vertical pressure available for adhesion and of tangential driving effort at the rails, greatest rocking couple and swaying couple, and greatest variation of vertical pressure on each rail (the so-called hammer-blow effect).

The half-angle between the balance masses, if fitted, is evidently

$$\tan^{-1} l/\mathrm{L} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

(18) **Example: Three Cranks at 120°.**—Fig. 11 shows the graphical solution for three equal masses, m, at equal radii r, with equal distances, l, between the cranks. Taking the axis OY in the plane of one of the cranks, the mr vector-polygon is closed, the mrl link-polygon in the plane OY is closed, but the mrl link-polygon in the plane OX is not closed, showing there is a resultant inertia couple in the plane OX. Two equal balance masses M_1 and M_2 in the planes of revolution Q_1 and Q_2 respectively will effect a balance. The value of the mrl of the resultant inertia couple, if left unbalanced, can be easily calculated as follows: The triangles 133′ and $\mathrm{O}c''a''$ are similar, therefore

$$\frac{3'3}{13'} = \frac{a''c''}{\mathrm{O}a''} \quad i.e.\ 3'3 \times \mathrm{O}a'' = 13' \times a''c''.$$

But $3'3 \times \mathrm{O}a''$ is the value of mrl for the resultant couple,

$$13' = 2l, \quad \text{and} \quad a''c'' = \frac{\sqrt{3}}{2}mr$$

$$\therefore mrl \text{ of resultant couple} = \sqrt{3}mrl \quad . \quad . \quad . \quad (7)$$

(19) **Example: Six Cranks at 120°.**—In a six-cylinder petrol engine, the cranks are arranged as two sets, shown in Fig. 11; the resultant couple of one set being equal and opposite to that of the other set. Numbering the cranks in order 1, 2, 3, 4, 5, 6 in the longitudinal elevation, in the end elevation the cranks overlap in pairs 1, 6; 2, 5; 3, 4. The

mr and both *mrl* polygons are closed, and therefore the balance is perfect. The student may draw the diagrams as an exercise.

(20) **General Conditions for Balance of Revolving Masses.**—From the discussion in sects. 10–15, the conditions

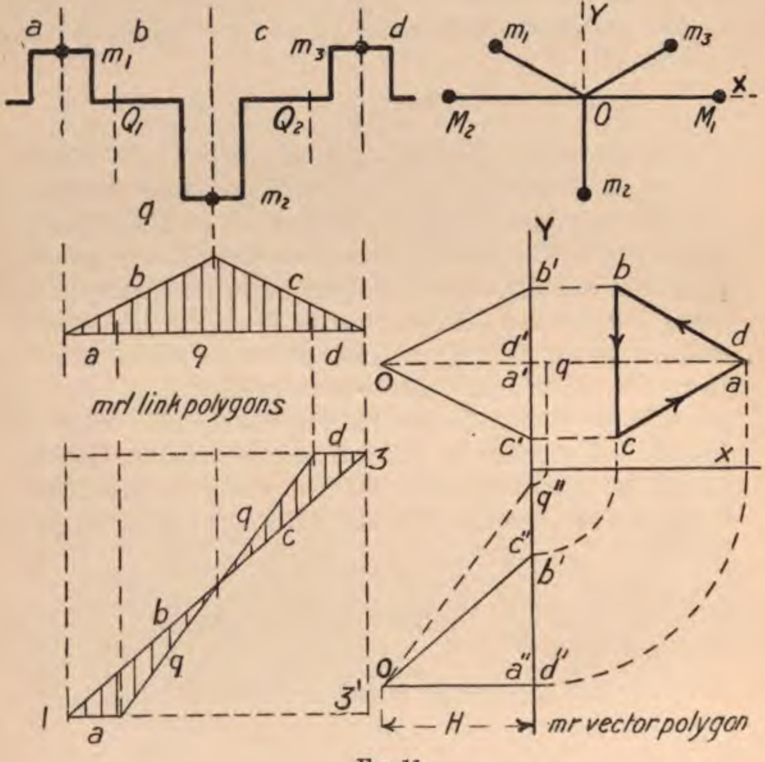

FIG. 11.

to be fulfilled in order that a number of revolving masses should be balanced among themselves, may be stated in either of three different forms, thus—

A. (1) The *mr* vector-polygon must be closed, and (2) the *mrl* vector-polygon with reference to any one transverse plane must be closed.

B. The *mrl* link-polygons in two distinct axial planes must be closed.

C. The *mrl* vector-polygons with reference to two distinct transverse planes must be closed.

From the discussion in sects. 10–15, and illustrated in Fig. 8, it is evident that a number of revolving masses in given transverse planes may be balanced among themselves, provided we are at liberty to assign or calculate the two crank-angles and the two *mr*'s for two of the masses; that is, four items in all, the other data being given arbitrarily or definitely. More generally, the data as to distance between planes of revolution, crank-angles and *mr*'s, may be all given arbitrarily, with the exception of four items; which must depend on, and be calculated from, the given data so as to satisfy the conditions of balance.

(21) **Number of Variables required to specify** n **Masses.**—To specify the magnitude and position of the inertia force due to one mass, three items, or variables, are required, viz. (1) the mass moment m_1r_1, or the equivalent mass M_1 reduced to a fixed radius R (sect. 1); (2) the distance l_1 of its plane of revolution from any transverse plane of reference; and (3) the angle θ_1 which the radius r_1 makes with an axial plane of reference revolving with the shaft. There are, therefore, apparently $3n$ variables required to specify n masses. But the *mr* vector-polygon will not be altered, except in size, if the various *mr*'s be multiplied or divided by a common factor; in other words, their various ratios determine the shape of the *mr* vector-polygon, if the crank angles are given. If we exclude the possible case of all the *mr*'s being zero, one of them must be given. Therefore, the independent *mr* variables are $(n - 1)$ in number. Taking the position of one crank as reference line, $(n - 1)$ angles define the positions of all the cranks, thus the independent crank-angle variables are $(n - 1)$ in number. Taking one plane of revolution as reference, the others are determined by $(n - 1)$ distances. But the two *mrl* link-polygons, or the *mrl* vector-polygons, are not altered in shape if the various distances between the transverse planes be multiplied or divided by a constant factor. The $(n - 1)$ distances determine $(n - 2)$ ratios, therefore $(n - 2)$ is the

number of independent variables defining the relative positions of the planes of revolution. Assuming one of the distances to be given, we thus exclude the possible case of all the distances being zero, which would mean that all cranks were in the same plane of revolution. Thus the total number of variables concerned in the problem of n revolving masses is $(3n - 4)$; one value of mr, one crank position and one distance between two planes of revolution, being taken to serve as datum to which the other $(3n - 4)$ variables are referred. If the n masses are to be in balance, 4 items must be left to be determined, and $(3n - 8)$ items, or $(3n - 8)$ relations between the variables, may be given arbitrarily. Of the 4 variables to be determined, not more than $(n - 1)$ should relate to the mr's, not more than $(n - 1)$ to the crank angles, and not more than $(n - 2)$ to the distances between the planes of revolution.

(22) **Example: Two Masses.**—From sect. 2, Chap. II., it is evident that two revolving masses can only be in balance if they are in the same plane of revolution, at an angle of 180° with each other.

Three Revolving Masses in Balance.—The number of variables that can be given arbitrarily is $3 \times 3 - 8 = 1$. That is, besides one mass-moment, one crank direction, and one distance between two planes of revolution, we can select only one other variable arbitrarily. Drawing the *mrl* vector-polygon (Fig. 12), with reference to the plane of revolution of the mass m_1, it reduces to two equal and opposite vectors CA and AB, the vector BC or $m_1r_1l_1$ being zero. Therefore algebraically

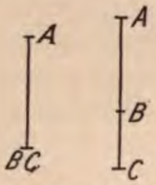

FIG. 12.

$$m_2r_2l_2 + m_3r_3l_3 = 0 \quad . \quad . \quad . \quad . \quad (1)$$

The sides of the *mr* vector-triangle CAB are parallel to those of the *mrl* vector-polygon. Therefore algebraically

$$m_1r_1 + m_2r_2 + m_3r_3 = 0 \quad . \quad . \quad . \quad . \quad (2)$$

and the three masses must lie in the same axial plane.

If the given arbitrary variable be either an mr or an l, equations (1) and (2) serve to determine the remaining variables.

If the given arbitrary variable is an angle between two cranks 1 and 2, the two vector-triangles being parallel, the sides BC are each of zero length, and B and C coincide in the mr vector-triangle. That is

$$\widehat{m_1r_1} = 0 \quad \text{and} \quad \widehat{m_2r_2} = -\widehat{m_3r_3} \quad . \quad . \quad (3)$$

From (1) and (3), it follows that $l_3 = l_2$; that is, the planes of revolution of m_2 and m_3 are identical. The solution with the given data, reducing masses to equal radii, therefore requires that m_1 is zero, $m_2 = m_3$, and the masses m_2 and m_3 are in the same plane of revolution, at an angle of 180°.

Exercises.

1. Two masses 12 and 18 lbs. revolving in the same plane at radii 30 and 10 inches respectively, inclined at an angle of 120° to each other, are to be balanced by a mass of 20 lbs. Find the position of the latter.
Ans. Radius 15·09 ins. at 90° with 10-inch radius.

2. To a rotating disc, 6 masses of 8, 20, 8, 16, 10, 24 lbs. are fastened all at the same radius and at equal angles. Find the magnitude and position of a mass fastened at the same radius which will balance the system.
Ans. 7·21 lbs. between 24 and 8 lb. masses, at 46·2° with the latter.

3. The crank and crank-pin of an engine are equivalent to a mass of 200 lbs. at 12 inches radius, the bearings are 28 inches apart centre to centre. Find the pressure on the bearings due to inertia, at a speed of 300 revolutions per minute. *Ans.* 3064 lbs. on each.

4. In example 3, balance masses are fastened, one to an extension of one of the crank cheeks 8 inches from the plane of motion of the connecting-rod, the other to a driving-pulley 20 inches from the plane of motion. The radius of the former is 20 inches, of the latter 30 inches. Determine the masses (*a*) when on opposite sides of the connecting-rod, (*b*) when on the same side. *Ans.* (*a*) 85·7 and 22·8 lbs. (*b*) 212·5 and 53·3 lbs.

5. Two cranks at right angles are 12 inches radius, 20 inches between planes of motion of connecting-rods, the equivalent revolving masses at the crank-pins are each 500 lbs. Find the magnitude and relative position of balance masses each at 30 inches radius, in planes 5 feet apart symmetrically disposed relative to the two cranks.
Ans. 149·1 lbs., 37° between balance masses.

6. Four cranks of equal radius 10 inches are at equal axial distances 15 inches along the crank-shaft. The middle pair are at 90°, and have each a mass of 100 lbs. at the crank-pin. Determine the masses and angles of the outside pair, so that the four masses may be self-balanced.

Ans. 74·5 lbs., 37° between outside pair.

7. Five cranks of equal radius 10 inches are at equal axial distances 15 inches along the crank-shaft. The middle three are at angles of 72°, and have each a mass of 100 lbs. at the crank-pin. Find the masses and angles of the outside cranks, so that the five masses may be self-balanced.

Ans. 94·0 lbs., $138\frac{1}{2}$° between cranks 1 and 2.

8. In example 7, if the three middle cranks are at 120°, find the masses and angles of the outside cranks.

Ans. 43·3 lbs., 150° between cranks 1 and 2.

9. In examples 3, 4, 5, 6, and 7, draw curves showing the relative bending moments at different parts of the shafts, and state the amount of the greatest bending moment on each shaft for an engine speed of 150 revolutions per minute.

Ans. (3) 10,724 lb.-in.; (4*a*) 8745 lb.-in.; (4*b*) 12,250 lb.-in.; (5) 57,080 lb.-in.; (6) 7195 lb.-in.; (7) 18,441 lb.-in.

10. Two masses of 10 lbs. and 20 lbs. respectively, are attached to a balanced disc at an angular distance apart of 90° and at radii of 2 feet and 3 feet respectively. Find the resultant force on the axis when the disc is making 200 turns a minute; and determine the angular position and magnitude of a mass placed at 2·5 feet radius which will make the force on the axis zero at all speeds. (Inst. C.E., Oct. 1903.)

Ans. 866 lbs. weight, 25·3 lbs., at $108\frac{1}{2}$° with 10 lbs. mass.

11. A shaft, 10 feet span between the bearings, carries two weights of 10 lbs. acting at the extremities of arms $1\frac{1}{2}$ feet and 2 feet long respectively; the planes in which the weights rotate being 4 feet and 8 feet respectively from the left-hand bearing, and the angle between the arms 60°. If the speed of rotation be 100 revolutions per minute, find the displacing forces on the bearings. Moreover, if the weights are balanced by two additional rotating weights, each acting at a radius of 1 foot, and placed in planes 1 foot from each bearing respectively, estimate the magnitudes of the two balance weights and the angles at which they must be set relative to the two arms. (Inst. C.E., Feb. 1904.)

Ans. Forces = 39·27 and 67·04 lbs. weight, masses 10·84 and 20·89 lbs., at angles 168° 20′, and 163° 50′, with $1\frac{1}{2}$ ft. and 2 ft. arms.

12. In the previous example, determine in both cases the greatest bending moment on the shaft. *Ans.* 2056, 1770 lb.-in.

13. In an inside-cylinder locomotive, the stroke is 26 inches, the distance between centres of cylinders is 23 inches, the revolving masses at each crank-pin which have to be balanced 350 lbs., the distance between the planes of revolution of the two balance masses in the driving-wheels

is 4 feet 10 inches. If the balance masses are at 2 ft. 6 in. radius, determine their magnitude and angular position relative to the cranks.

Ans. 200 lbs. at 17·3 in. radius, 155° 40′ with crank.

14. In an outside-cylinder locomotive, the distance between centres of cylinders is 6 feet 1 inch, the revolving masses at the crank-pins which have to be balanced are each 250 lbs.; the rest of the data is as in example 13. Determine the positions and magnitude of the balance weights. *Ans.* 200 lbs. at 18·47 in. radius, 186° 30′ with crank-pin.

15. In a shaft with three cranks at 120°, the crank-pin radius is 10 inches, distance between adjacent cylinders 30 inches, the mass of each crank, reduced to crank-pin radius, is 150 lbs. Find the magnitude and relative angular position of two masses placed at 12 inches radius, and at 75 inches between their planes of revolution, so that there may be no pressure on the bearings due to inertia forces.

Ans. 86·6 lbs., at 150° with nearest crank.

16. In the three-crank shaft of example 15, the two outside cranks are overhung, and there are no balance weights. Find the pressure on the two bearings which are 40 inches apart centre to centre, at a speed of 200 revolutions per minute. *Ans.* 2210 lbs.

CHAPTER IV

MASSES RECIPROCATING IN THE SAME AXIAL PLANE; LONG CONNECTING-RODS

In this chapter will be discussed the inertia forces due to the reciprocating masses of an engine, having the axes of its cylinders all in the same axial plane, or longitudinal plane passing through the axis of the crank-shaft. This embraces most examples of engines having two or more cylinders and cranks, *e.g.* the usual types of locomotive engine, marine engine, motor-car engine, etc. The discussion in this chapter does not take any account of the obliquity of the connecting-rod; *i.e.* the length of the connecting-rod is assumed infinite. The influence of the obliquity of the connecting-rod is discussed in Chapter VI.

(1) **Simple Harmonic Motion.**—Let P (Fig. 1) be a point moving with uniform speed v in a circle of radius r. Draw any diameter X_1OX of the circle, and draw Pp at right angles to OX. As P moves in the circle, p moves to and fro along the straight line X_1OX. If the linear speed of P is constant, p is said to move with *simple harmonic motion* (S.H.M.). The *period*, or the *periodic time*, T, of the simple harmonic motion, is the time taken by P to make one revolution.

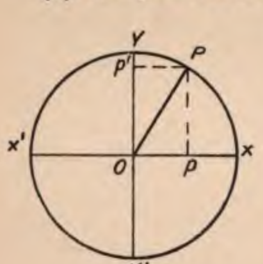

Fig. 1.

In sect. 1, Chap. II., it has been shown that the acceleration

of P is in the direction PO, and is of magnitude $\omega^2 r$. Therefore, by choosing a suitable scale, the acceleration of P may be represented by the vector $\hat{\mathrm{PO}}$. From sect. 8, Chap. I., the acceleration of p will be represented to the same scale by the vector $\hat{p\mathrm{O}}$. Hence, in simple harmonic motion, the acceleration is always directed towards the centre, and *is proportional to the displacement from the centre.* When P is at X, p and P coincide. Therefore, the maximum acceleration of p is equal to that of P, *i.e.* $\omega^2 r$. See *eq* (2) Chap. II.

If mr is the mass of the body at p moving with simple harmonic motion, the force acting on it to produce the motion is $\omega^2 m \times \hat{p\mathrm{O}}$. The maximum value of the force occurs when p is at the end of its stroke, and is $mr\omega^2$, or in gravitation units, $\mathrm{F} = mr\omega^2/g$ . . . . (1)

A heavy mass, m, suspended by a long spiral spring, hanging vertically, if displaced from the position of rest, oscillates above and below its position of rest. The stretch of the spring is proportional to the force applied, and conversely the pull on the lower end of the spring is proportional to its total extension. When the mass is at its position of rest, the pull P on the spring is exactly equal to the weight W of the mass, and the resultant force $\mathrm{F} = (\mathrm{P} - \mathrm{W})$ acting on the mass is zero. When the mass is in any other position, the resultant force $(\mathrm{P} - \mathrm{W})$ is proportional to the displacement from the position of rest. Therefore, the motion is simple harmonic, neglecting frictional resistances. For this case, equation (1) may be put in a more convenient form. Let T be the periodic time; that is, the time of one complete up-and-down oscillation, and therefore also the time of one revolution of the point P (Fig. 1). The angle of one complete revolution is equal to the angular speed multiplied by the time; that is

$$2\pi = \omega \mathrm{T}, \quad \text{or} \quad \omega = 2\pi/\mathrm{T} \quad . \quad . \quad . \quad (2)$$

(1) may be written

$$\omega = \sqrt{\frac{\mathrm{F}g}{mr}} = \frac{2\pi}{\mathrm{T}}$$

Therefore

$$\mathrm{T} = 2\pi\sqrt{\frac{m}{g}\cdot\frac{r}{\mathrm{F}}} \quad . \quad . \quad . \quad (3)$$

But r is the displacement, F the corresponding resultant force acting on the mass, therefore r/F is constant. The periodic time is therefore independent of the amplitude of the oscillation.

If the crank-shaft of an engine revolves uniformly, the piston moves approximately with simple harmonic motion; the deviation from simple harmonic motion being due to the obliquity of the connecting-rod. The greater the ratio of connecting-rod length to stroke, the less is the deviation from simple harmonic motion. In this chapter, the reciprocating parts of an engine will be considered as moving with simple harmonic motion. The inertia forces due to this simple harmonic motion are called the "primary" inertia forces, and the balancing of them may be called "primary balancing." The influence of the obliquity of the connecting-rod is discussed in Chapter VI.

(2) **Uniform Motion in a Circle Equivalent to Two S.H.M.'s at Right Angles.**—In Fig. 1, draw a second diameter Y_1OY at right angles to X_1OX, and draw Pp' at right angles to OY. The motion of p' along Y_1OY is simple harmonic. The two simple harmonic motions of p and p', differ only in *phase;* that of p' being later than that of p by the time of a quarter revolution of P, *i.e.* $\frac{1}{4}$T.

The uniform circular motion of a mass m at P, is equivalent to the simple harmonic motions of two equal masses m at p and p' respectively. For vector $\widehat{PO}$ = vector $\widehat{po}$ + vector $\widehat{p'o}$. (Sect. 11, Chap. I.)

From the above, it is readily seen that a single-cylinder engine cannot be balanced by any disposition of counterbalance masses fixed to, and revolving with, the crank-shaft. The piston, piston-rod, crosshead, etc., move approximately with simple harmonic motion along the line of stroke. By suitably fixing counterbalance masses to the crank-shaft the inertia forces along the line of stroke due to the piston, etc., may be balanced, completely or partially; but, in doing so, inertia forces in a direction at right angles to the plane passing through the axes of the shaft and cylinder are introduced.

Similarly, in a locomotive with two cylinders exactly horizontal, the inertia forces due to the pistons, etc., have no vertical component; they produce no "hammer blow" on the rails. If the two cranks were parallel, and symmetrically situated with respect to the central vertical longitudinal plane, the inertia forces would merely produce an alternating accelerating and retarding effect on the frame of the engine. But the cranks being set at right angles, for ease of starting and uniformity of driving effort, the inertia forces of the two pistons are not in the same phase, and produce an alternating couple in a horizontal plane, tending to derail the engine. The horizontal inertia forces may be balanced partially or completely by counterbalance masses rotating with the crank-shaft, thus reducing or eliminating the derailing tendency; at the expense, however, of introducing vertical inertia forces giving hammer blows on the rail. This is further discussed in Chapter V.

(3) **Simple Harmonic Motion equivalent to Two Uniform Circular Motions in Opposite Directions.**—Let OP be the crank in any position, PC the connecting-rod, COX the line of stroke (Fig. 2). With C as centre and radius CP

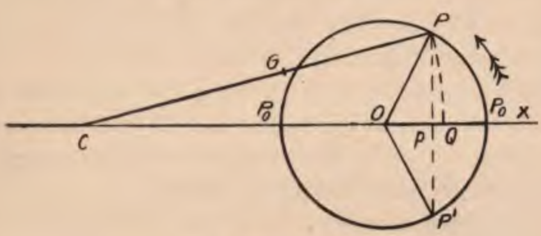

FIG. 2.

draw an arc cutting the line of stroke at Q. Let m be the mass of the parts rigidly connected to, and moving with, the crosshead C. The mass m may be considered as transferred to the point Q, since the accelerations, forces, etc., for the mass m at Q are exactly the same as for the mass m at C, or at any other point on the centre line of the piston-rod.

Draw Pp perpendicular to OX, meeting OX at p, and

produce Pp to cut the crank-pin circle again at P′. The inertia forces of a mass m at p are the same as those for two masses each $\frac{1}{2}m$ at P and P′ respectively; since p is their mass-centre. Therefore, the inertia forces due to the simple harmonic motion of the piston masses m reciprocating along the line of stroke COX, are the same as those due to a mass $\frac{1}{2}m$ fixed to and moving with the real crank-pin, and a mass $\frac{1}{2}m$ fixed to an imaginary crank-pin revolving in the reverse direction; the real and imaginary cranks coinciding with each other on the line of stroke, *i.e.* when the crank is on the dead centre.

(4) **Reciprocating Masses in the Same Axial Plane.**—The inertia forces at any instant due to a number of masses driven by the same revolving shaft and reciprocating in the same axial plane all lie in that plane. The resultant force and resultant couple at any instant must therefore also lie in that plane. The simple harmonic motion of each reciprocating mass is the projection on the longitudinal plane of an equal mass fixed to and revolving with the corresponding crank-pin. The resultant force and resultant couple due to the revolving masses at the crank-pins are each represented by a vector of constant magnitude, rotating with the shaft, and are found by the methods of Chap. III.; their projections at any instant on the axial plane give the resultant force and resultant couple for the given reciprocating masses.

If the given reciprocating masses are balanced among themselves, the mr and the two mrl polygons for the equivalent revolving masses at the crank-pins must be closed. If not balanced among themselves, balance may be effected by two reciprocating masses at any two transverse planes Q_1 and Q_2, their mr's and the corresponding crank-angles for driving the two reciprocating masses being determined by the method of Fig. 8, Chap. III. Such reciprocating balance masses, or "bob-weights," have been used by Mr. Yarrow to balance the engines of a torpedo-boat; but the expense and complication involved render this method of balancing an engine, in general, impracticable.

The student must carefully note that the balance cannot

usually be effected by merely fixing *revolving* masses in the planes of revolution Q_1 and Q_2. All the forces in the longitudinal plane of the given reciprocating masses may be balanced thereby, but unbalanced forces in an axial plane at right angles to that of the reciprocating masses would be introduced.

If the reciprocating masses are unbalanced, the resulting disturbance can be expressed as due to two reciprocating *mr* vectors in two transverse planes chosen at random, the positions of the two rotating *mr* vectors of which they are projections being determined by the methods of Chap. III. In a locomotive, by choosing these two planes as the planes of contact of the driving-wheels with the rails, the variation of effort at each rail can be estimated.

Example.—In a four-cylinder engine, given the distances between the cranks, the radii of and angle between two of the cranks, and their reciprocating masses, find the radii, and angles of the remaining two cranks and their reciprocating masses, so that the system may be in primary balance.

Replacing the reciprocating masses by equal revolving masses at the crank-pins, the final *mr* and *mrl* polygons must be closed. Let 1 and 2 (Fig. 3) be given cranks, Q_1 and Q_2 the cranks whose angles and masses have to be determined. Draw the *mr* vectors $\widehat{ab}$ and $\widehat{bc}$ for the given cranks. Draw OY and OX respectively parallel and perpendicular to $\widehat{ac}$, and project the corners a, b, c, on OY and OX. Rotate OY and OX into positions OY′ and OX′ respectively; choose the poles O′. Draw the sides a, b, and c of the two *mrl* link-polygons, thus determining their closing sides q. Draw the radius vectors Oq' and Oq'', and rotate and project q' and q'', determining the point q in the *mr* vector-polygon. $\widehat{qa}$ and $\widehat{cq}$ are the vectors M_1R_1 and M_2R_2 for the cranks at Q_1 and Q_2 respectively.

(5) **Number of Variables for Self-balancing.**—Since a mass reciprocating with simple harmonic motion is specified completely by stating the mass, radius of equivalent crank, crank angle, and plane of revolution, exactly like a revolving

mass, the number of variables required to specify n reciprocating masses is $(3n - 4)$. (See sect. 21, Chap. III.) If they are in balance, $(3n - 8)$ variables may be assigned arbitrarily,

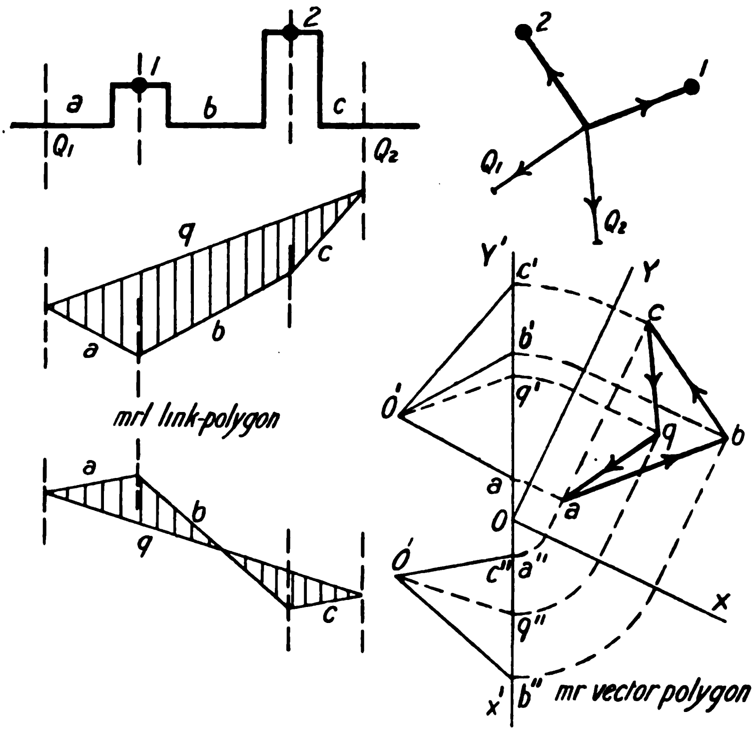

Fig. 3.

leaving 4 to be determined, of which not more than $(n - 1)$ should relate to the mr's, not more than $(n - 1)$ to the crank angles, and not more than $(n - 2)$ to the distances between the lines of reciprocation.

Exercises.

1. The piston of a petrol engine weighs 3 lbs., the stroke is 4 inches. At a speed of 2000 revolutions per minute, what force is required to accelerate the piston at the beginning of its stroke, neglecting the obliquity of the connecting-rod? *Ans.* 1360 lbs. weight.

2. A long spiral spring which stretches one inch for each 10 lbs. of

load, hangs vertically and is loaded with 100 lbs. Find its periodic time of oscillation, neglecting the mass of the spring. *Ans.* 0·71 second.

3. Define *simple harmonic motion*, and show that in such motion the acceleration is proportional to the distance from the middle of the path. A body weighing 10 lbs. suspended from a spring makes a complete oscillation up and down in two seconds. How much will the same spring be stretched by a 1 lb. weight? (Inst. C.E., Oct. 1897.)
Ans. 3·9 inches.

4. In a four-crank engine with equal crank radii, the cranks are at equal axial distances. The angle between the middle pair of cranks is 90°, the reciprocating masses are 200 lbs. and 300 lbs. respectively. Determine the crank angles and the reciprocating masses of the outside cranks for self-balancing, neglecting obliquity of connecting-rods.
Ans. 167 and 209 lbs., 142° and 160½° with 200 and 300 lbs.

5. In a three-crank engine, cranks at 120°, the cranks are 18 inches radius, distance between cylinders 7 feet 6 inches, reciprocating masses each 3 tons. Find the unbalanced couple at a speed of 150 revolutions per minute. *Ans.* 448 ton-inch.

6. Show that when a helical spring vibrates freely under the action of a weight, its periodic time is the same as that of a simple pendulum having a length equal to the static extension of the spring when carrying the weight—the mass of the spring itself being neglected. (Inst. C.E., Oct. 1903.)

7. If in the locomotives of examples 13 and 14, Chap. III., the driving-wheels are 7 feet diameter, reciprocating mass for each cylinder is 400 lbs., distance between rail centres 5 ft., and no balance masses are fitted, determine what would be the greatest disturbing force at each rail, at a speed of 50 miles per hour. What would then be the greatest magnitude of the swaying couple?
Ans. 4750, 6534 lbs.; 590, 2538 lb.-ft.

8. In the locomotives of example 7, if the distance between the axle bearings is 3 feet 9 inches, centre to centre, determine the greatest horizontal pressure due to inertia at the speed of 50 miles per hour.
Ans. 4672, 7915 lbs. weight.

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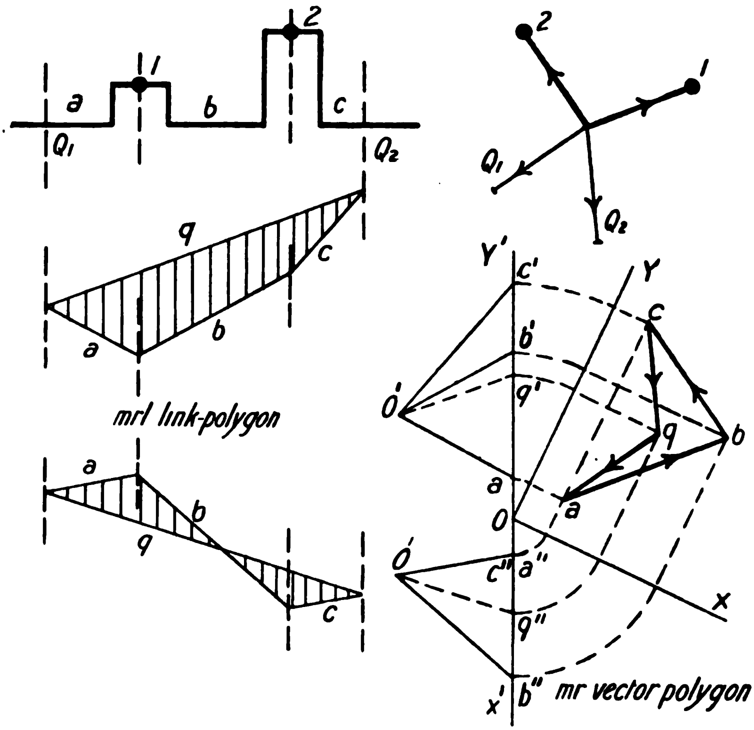

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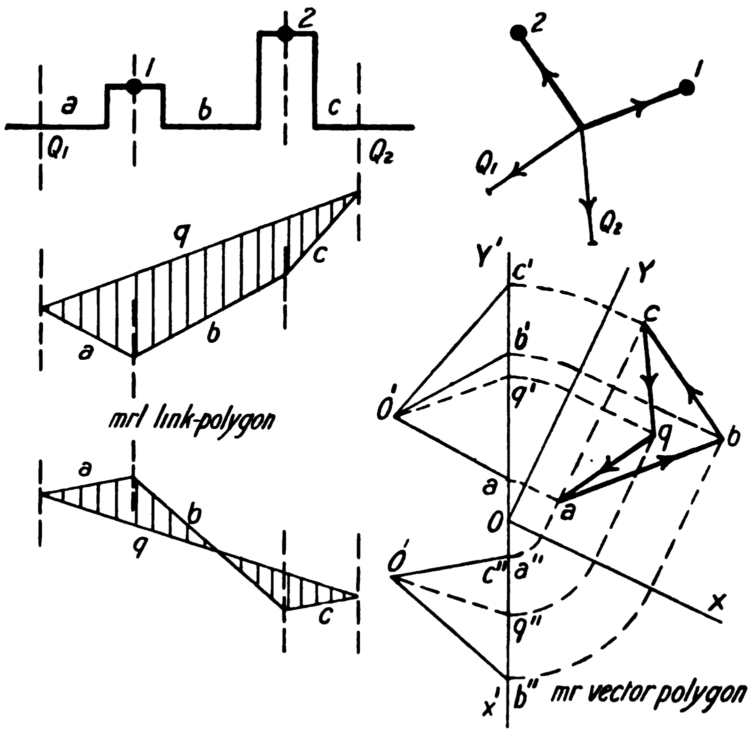

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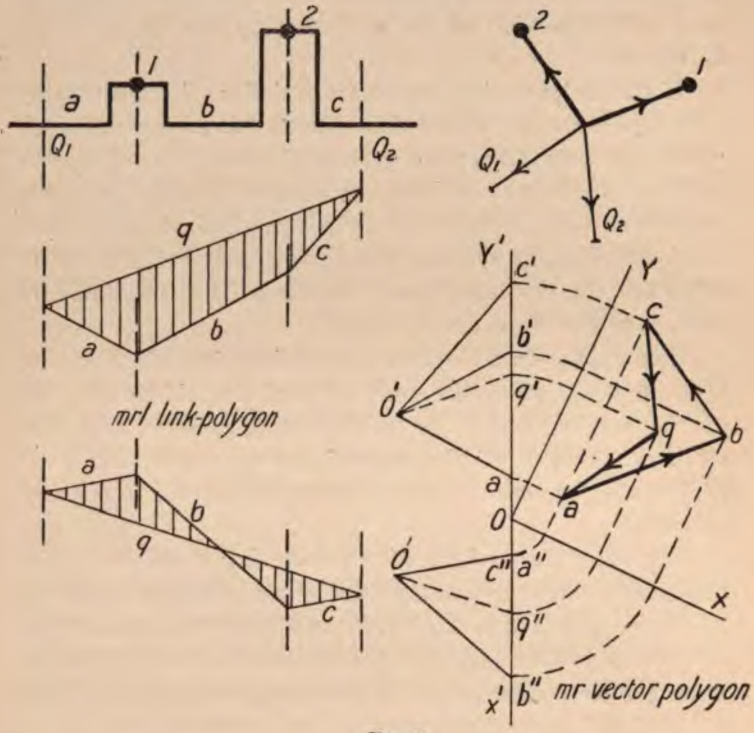

Fig. 3.

leaving 4 to be determined, of which not more than $(n - 1)$ should relate to the mr's, not more than $(n - 1)$ to the crank angles, and not more than $(n - 2)$ to the distances between the lines of reciprocation.

Exercises.

1. The piston of a petrol engine weighs 3 lbs., the stroke is 4 inches. At a speed of 2000 revolutions per minute, what force is required to accelerate the piston at the beginning of its stroke, neglecting the obliquity of the connecting-rod? *Ans.* 1360 lbs. weight.

2. A long spiral spring which stretches one inch for each 10 lbs. of

load, hangs vertically and is loaded with 100 lbs. Find its periodic time of oscillation, neglecting the mass of the spring. *Ans.* 0·71 second.

3. Define *simple harmonic motion*, and show that in such motion the acceleration is proportional to the distance from the middle of the path. A body weighing 10 lbs. suspended from a spring makes a complete oscillation up and down in two seconds. How much will the same spring be stretched by a 1 lb. weight? (Inst. C.E., Oct. 1897.)

Ans. 3·9 inches.

4. In a four-crank engine with equal crank radii, the cranks are at equal axial distances. The angle between the middle pair of cranks is 90°, the reciprocating masses are 200 lbs. and 300 lbs. respectively. Determine the crank angles and the reciprocating masses of the outside cranks for self-balancing, neglecting obliquity of connecting-rods.

Ans. 167 and 209 lbs., 142° and 160½° with 200 and 300 lbs.

5. In a three-crank engine, cranks at 120°, the cranks are 18 inches radius, distance between cylinders 7 feet 6 inches, reciprocating masses each 3 tons. Find the unbalanced couple at a speed of 150 revolutions per minute. *Ans.* 448 ton-inch.

6. Show that when a helical spring vibrates freely under the action of a weight, its periodic time is the same as that of a simple pendulum having a length equal to the static extension of the spring when carrying the weight—the mass of the spring itself being neglected. (Inst. C.E., Oct. 1903.)

7. If in the locomotives of examples 13 and 14, Chap. III., the driving-wheels are 7 feet diameter, reciprocating mass for each cylinder is 400 lbs., distance between rail centres 5 ft., and no balance masses are fitted, determine what would be the greatest disturbing force at each rail, at a speed of 50 miles per hour. What would then be the greatest magnitude of the swaying couple?

Ans. 4750, 6534 lbs.; 590, 2538 lb.-ft.

8. In the locomotives of example 7, if the distance between the axle bearings is 3 feet 9 inches, centre to centre, determine the greatest horizontal pressure due to inertia at the speed of 50 miles per hour.

Ans. 4672, 7915 lbs. weight.

CHAPTER V

REVOLVING AND RECIPROCATING MASSES

(1) **Motion of the Mass-centre of the Connecting-rod.**—Let G (Fig. 2), Chap. IV., be the mass-centre of the connecting-rod, and m_g its mass. The motion of the connecting-rod may be resolved into a motion of translation of its mass-centre G, together with an angular motion of oscillation about an axis through G, perpendicular to the plane of motion of the rod. The system of forces required to produce the motion of the connecting-rod may be resolved into two sets, the first set those required to produce the motion of the mass-centre G, the second set those required to produce the angular swing about the mass-centre G. The latter set of forces at any instant constitute a couple in the plane of motion of the rod, that is, the axis of the couple is parallel to the axis of the crank-shaft. It may be called the "transverse couple," and is considered in Chap. VII.

The first set of forces are the same as required for two masses, m_c and m_p, at the cross-head and crank-pin respectively, having the total mass m_g, and having their joint mass-centre at G. That is—

$$m_c + m_p = m_g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$m_c = \frac{\text{GP}}{\text{CP}} m_g \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$m_p = \frac{\text{CG}}{\text{CP}} m_g \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Therefore, the linear motion of the connecting-rod is taken into account, by dividing its mass into a revolving

mass fixed to the crank-pin, and a reciprocating mass moving with the crosshead.

(2) **Engine balanced by Masses on Connecting-rod and Crank-shaft.**—Mr. Wilson Pilcher has designed an engine, in which the balance is effected by balance weights attached beyond the big end of the connecting-rod. If the mass-centre G of the connecting-rod and balance-weight lie beyond P in CP produced (Fig. 2, Chap. IV.), m_c is negative. If the balance masses be disposed so that ^-m_c is equal to the reciprocating masses moving with the cross-head, the algebraic sum of the masses at the cross-head is zero. The mass m_p at the crank-pin can be balanced by a mass fixed to and revolving with the crank-pin. The engine is therefore perfectly balanced, except as regards the transverse couple due to the angular swing of the connecting-rod. Unfortunately, the balance mass to be added to the connecting-rod considerably increases this transverse couple (see Chap. VII.). If, however, an engine with two cranks at 180° be made on this design (Fig. 1), (*i.e.* with each piston balanced, as

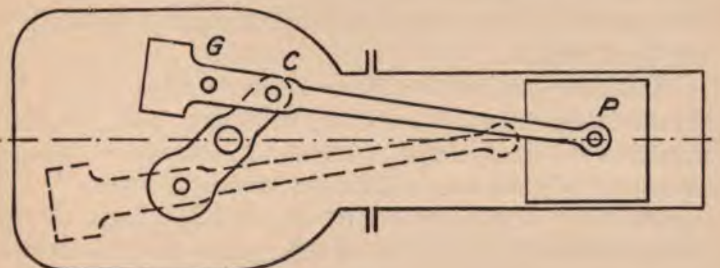

FIG. 1.

above described, by a mass fixed beyond the big end of the connecting-rod), the angular accelerations of the two connecting-rods at any instant are equal and opposite (see Chapter VII.), and the transverse couples are in balance. Hence this engine is mathematically in perfect balance.

(3) **Reciprocating Masses equivalent to "Forward" and "Reverse" Revolving Masses.**—From the theorem of sect. 3, Chap. IV., it is evident that in an engine with any

number of cylinders the inertia forces of the reciprocating masses can be dealt with by replacing them by a set of masses fixed to the various crank-pins and revolving with the crank-shaft, and a set of masses fixed to imaginary cranks revolving in the reverse direction. They may be called the equivalent "forward" and "reverse" masses respectively. It has been shown, in Chap. III., that the equivalent forward masses can always be balanced by two masses fixed to the shaft. It is, however, impracticable to arrange counterbalance masses revolving in the reverse direction to that of the crank-shaft; hence, if the equivalent reverse masses give rise to a resultant force or a resultant couple, the primary balance cannot be made perfect.

Give any number of masses fixed to and revolving with a shaft, and any number of reciprocating masses driven with simple harmonic motion from the shaft, the problem of finding the resultant inertia force and couple, reduces to finding the resultant force and couple for the actual and equivalent forward masses, and for the equivalent reverse masses respectively; then finding the final resultant of the two resultant forces, and the final resultant of the two resultant couples.

If in any engine the forward masses are balanced by counterbalance masses revolving with the shaft, the final resultant force and couple are each of constant magnitude, and revolve in the reverse direction at the same speed as the crank-shaft. Any alteration of the counterbalance masses from the disposition required to balance the forward masses, will make the final resultant force and the final resultant couple greater at one position and less at another position of the crank-shaft; the maximum and minimum values each occurring twice in a revolution, as shown in Fig. 5. Thus, in general, the balance may be said to be the best possible, when the equivalent forward masses are balanced. In some cases, however, *e.g.* a locomotive, a want of balance in one direction may be more objectionable than in another, and the equivalent forward masses may with advantage be *over* balanced in one case, *under* balanced in another case.

In a single-cylinder engine (Fig. 2, Chap. IV.), the equivalent forward mass is balanced by a mass $\frac{1}{2}m$ fixed opposite the crank-pin at a radius r. The engine may then be said to be "half balanced."

Let Fig. 2 be a longitudinal view of the crank-shaft showing the pitch of the cylinders or cranks, Fig. 3 an end view showing the angle between the various cranks from which the simple harmonic motions are derived, OV being the trace of the plane of reciprocating masses. Replacing each reciprocating mass m by an equivalent forward revolving mass $\frac{1}{2}m$ at the corresponding crank-pin, and an equivalent reverse revolving mass $\frac{1}{2}m$, the resultant

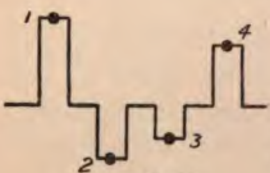

FIG. 2.

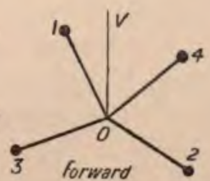

FIG. 3.

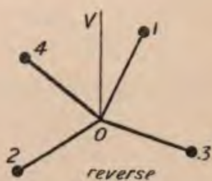

FIG. 4.

force and couple due to the equivalent forward masses are obtained by the procedure shown in Fig. 8, Chap. III.

Fig. 4 shows the positions of the equivalent reverse masses corresponding to the positions of the forward masses Fig. 3; the various forward and reverse masses making equal angles with OV, the line of stroke. Thus, Fig. 4 is an image V, with respect to OV, of Fig. 3.

Applying the method of Fig. 8, Chap. III., to the reverse masses of Fig. 4, the mr diagrams forward and reverse are images one of the other, with respect to OV. The mrl link-polygons in the plane OY are identical for the forward and reverse masses; the mrl diagrams in the plane OX for the forward and reverse masses only differ in the algebraic sign of the ordinates, that is, they are skew images, one of the other, with respect to the closing lines. Thus, the plane F of the reverse resultant force is the same as that of the forward resultant force, and the resultant MR vectors for the forward and reverse masses are of equal angles on opposite sides

of OV. From sect. 3, Chap. IV., the final resultant MR is in the plane OV. Similarly the final resultant MRL is in the plane OV.

This is also evident from first principles. Since the given masses all move in the same plane, their final resultant MR and MRL must be in the same plane. When only reciprocating masses in the same axial plane, and no revolving masses, are dealt with, they may be best considered as projections of revolving masses, and problems will be solved by the methods of Chaps. III. and IV. But where both revolving and reciprocating masses are connected with the shaft, it may be most convenient to replace the reciprocating masses by their equivalent forward and reverse revolving masses, as detailed in sect. 5.

(4) **Resultant of a Forward and Reverse Vector revolving at Same Speed.**—Let the same vectors $\hat{OA}$ and OB of constant length (Fig. 5) revolve with equal speeds in opposite directions. The resultant $\hat{OR}$ of $\hat{OA}$ and $\hat{OB}$ is found by completing the parallelogram AOBR. The maximum value $\hat{OR}^0$ evidently occurs when $\hat{OA}$ and $\hat{OB}$ coincide in direction OX, and is then equal to the sum of the lengths OA and OB. The minimum value $\hat{OR}_1$ occurs when $\hat{OA}$ and $\hat{OB}$ have each rotated through a right angle from OX. The locus of R is an ellipse, its major and minor semi-axes being $(a+b)$ and $(a-b)$ respectively, where a and b are the lengths of OA and OB respectively.

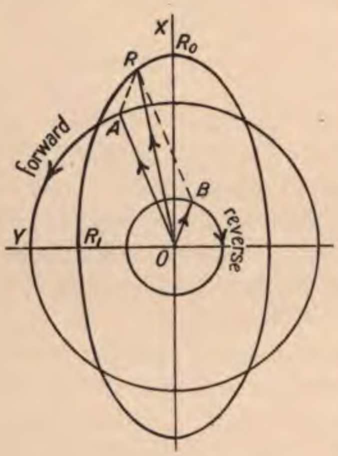

Fig. 5.

If OA = OB, the minor semi-axis $(a-b)$ is zero; *i.e.* the locus of R is along the straight line OX, as is evident

also from sect. 3. That is, R moves with simple harmonic motion along OX.

(5) **General Method of finding Unbalanced Force and Unbalanced Couple and Reactions at Bearings for any Number of Revolving and Reciprocating Masses.**— It has been shown in Chap. III., that the revolving masses can

Transverse planes	1	2	3	4	5	Q_1	Q_2
Distance from plane No. 1	0						
Angle of crank from crank No. 1	0°						
Revolving masses at crank-pins	300	500	300	500	300		
Reciprocating masses	400	500	0	500	0		
Connecting-rod masses— $\frac{1}{3}$ at crosshead, $\frac{2}{3}$ at crank-pin	240	420		300			
Combined revolving masses	300 160 480	500 280 780	300	500 200 700	300		
Combined reciprocating masses	400 80 480	500 140 640	0	500 100 600	0		
Equivalent combined forward masses	480 240 720	780 320 1100	300	700 300 1000	300		
Equivalent reverse masses	240	320	0	300	0		

always be balanced by suitably placing two balance masses; and in Chap. IV., that the reciprocating masses may be made

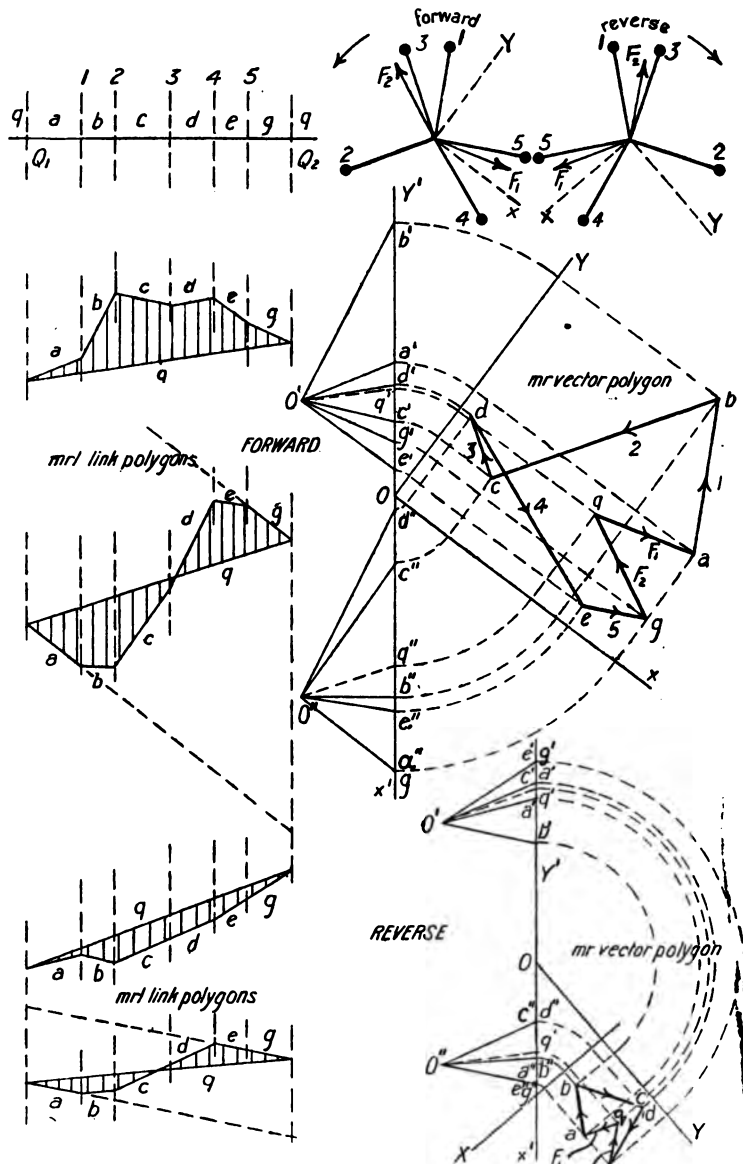
forward
reverse
mr vector polygon
mrl link polygons
FORWARD
REVERSE
mr vector polygon
mrl link polygons

FIG. 6.

to balance, provided we are at liberty to choose the position of two of the cranks, and the magnitudes of the reciprocating masses driven by them. In many cases, however, the crank angles are assigned from considerations of uniformity of driving torque, not from considerations of balancing, and both sets of masses may be unbalanced.

The masses of cranks, balance-weights, pistons, connecting-rods, etc., being given, they should be tabulated as in lines 3, 4, and 5 of the schedule on p. 71. The masses of the connecting-rods are divided in the proper ratio between the crank-pins and cross-heads, and the combined revolving and reciprocating masses tabulated (lines 6 and 7). The combined reciprocating masses are divided into equivalent forward and reverse revolving masses, and the equivalent forward and reverse masses tabulated (lines 8 and 9). If the crank radii are all equal, the *mr*'s will be proportional to the equivalent forward and reverse masses.

The distances between the transverse planes and the crank angles being given, the longitudinal- and crank-angle diagram is drawn (Fig. 6). Q_1 and Q_2 are taken as the planes of the two bearings. The *mr* vector-polygon for the forward masses is then drawn, the *mrl* link-polygons in the planes OY and OX, and the M_1R_1 and M_2R_2 for the "forward" reactions at Q_1 and Q_2, obtained by the methods of Chap. III.

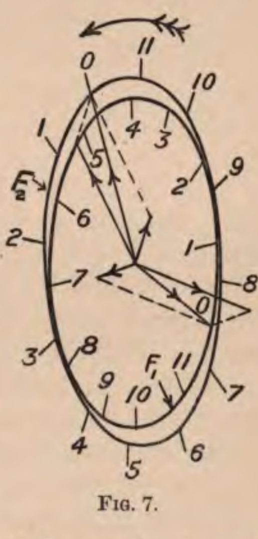

FIG. 7.

Similarly, the *mr* vector-polygon for the reverse masses, the *mrl* link-polygons in the planes OY and OX, and the M_1R_1 and M_2R_2 for the "reverse" reactions at Q_1 and Q_2 are successively obtained.

The forward and reverse M_1R_1's at Q_1 may then be combined.

The ellipse showing the resultant of the rotating forward and reverse M_1R_1 vectors at Q_1, may then be drawn by the

method shown in Fig. 5; similarly for the resultant of the M_2R_2 vectors at Q_2. These two ellipses are shown at F_1 and F_2 (Fig. 7), the point O on each ellipse being obtained from the corresponding forward and reverse F_1 and F_2 vectors shown in Fig. 6. The twelve points marked on each ellipse show the resultant vectors for equal angular intervals the crank-shaft has described from the position shown in Fig. 6.

(6) **Partial Balancing of Locomotives.**—In a locomotive it is desirable to express the vertical and horizontal disturbing forces and couples separately; the following method may then be used. Figs. 8 and 9 show the arrange-

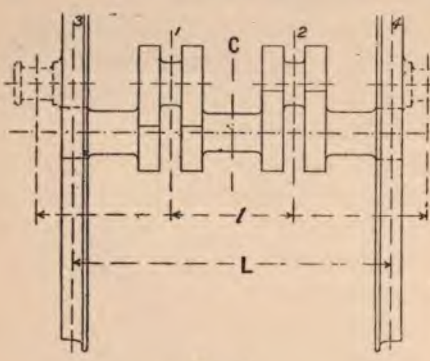

FIG. 8.

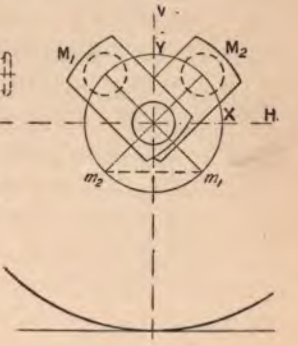

FIG. 9.

ment of the driving-axle of an inside-cylinder locomotive, L being the distance between the planes of contact of the driving-wheels with the rails, l is the distance between the planes 1 and 2 of the cranks, or centres of the cylinders, and r the crank-radius. In an outside-cylinder locomotive, l is greater than L, the crank-pins projecting from the outsides of the driving-wheels, as shown by the dotted lines. In Fig. 9, OH is the centre line of the cylinders (assumed to be horizontal in the following discussion), OV a line at right angles.

The revolving masses rotating with the driving-axle can be balanced by two balance masses in the driving-wheels, as shown in sect. 17, Chap. III., similarly for the coupling-rod masses, therefore these need not be further referred to

here. Let M be the reciprocating mass for each cylinder, which includes the mass of piston-rod, cross-head, and part of the mass of connecting-rod, as discussed in sect. 1. Let m be the portion of this reciprocating mass that has to be balanced. This can be done by imagining two balance masses each equal to m, fixed at m_1 and m_2 opposite the cranks M_1 and M_2, and lying respectively in the planes of revolution 1 and 2. The imaginary masses at m_1 and m_2 are equivalent to two actual balance masses which can be placed in the driving-wheels, their magnitudes and positions being determined by the method of sect. 17, Chap. III.

The imaginary revolving masses at m_1 and m_2, and therefore also the actual balance masses in the driving-wheels, are equivalent to two pairs of reciprocating masses, each equal to m, driven by the crank-pins m_1 and m_2, and reciprocating along OH and OV respectively. Thus, the masses reciprocating horizontally are equivalent to two masses, each $(M-m)$ driven by the actual crank-pins M_1 and M_2. The unbalanced force and couple are determined as in sect. 17, Chap. III. The horizontal unbalanced mr vector (which produces a variation of the horizontal effort on the whole train) is the projection on OH of an mr vector, rotating in the central plane C, and lying in the rotating plane OY midway between the cranks. The magnitude of the rotating vector is therefore—

$$\sqrt{2}(M - m)r \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, the horizontal unbalanced mrl vector, which produces the swaying couple, is the projection of a vector rotating in the plane OX, of magnitude—

$$\frac{1}{\sqrt{2}}(M - m)rl \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the same way, taking the masses reciprocating vertically, the vertical unbalanced mr vector, which produces the variation of vertical load of the driving-wheels on the rails, is the projection on OV of an mr vector rotating in the plane C and lying in the rotating plane OY, of magnitude—

$$\sqrt{2}mr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

This vector is equivalent to two equal vertical vectors in the planes 3 and 4, each the projection of a rotating vector of magnitude

$$\frac{1}{\sqrt{2}}mr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The vertical unbalanced mrl vector, which produces the transverse rocking couple, is the projection on OV of a rotating vector in the rotating plane OX, of magnitude

$$\frac{1}{\sqrt{2}}mrl \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This is equivalent to two equal opposite vertical mr vectors in the planes 3 and 4, each the projection of a rotating vector of magnitude

$$\frac{1}{\sqrt{2}}mr\frac{l}{\mathrm{L}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The directions of the rotating mr vectors (4) and (6) being at right angles, their resultant, which produces the variation of vertical pressure on each rail (the so-called hammer blow), is of magnitude

$$mr\sqrt{\tfrac{1}{2}\left\{1+\left(\frac{l}{\mathrm{L}}\right)^2\right\}} \quad . \quad . \quad . \quad . \quad (7)$$

The above expressions are similar in form to those of sect. 17, Chap. III. Multiplying them by ω^2/g, the maximum values of the disturbing forces or couples are obtained in gravitation units.

Data for Examples.

M = 400 lbs. for piston, piston-rod, and cross-head + 140 lbs. part mass of connecting-rod = 540 lbs.

Stroke 26 inches; r = 13 in. = 1·08 ft.

L = distance between centres of rail = 5 ft.

l, for inside cylinders, = 23 in. = 1·92 ft.

l, for outside cylinders, = 6·12 ft.

Diameter of driving-wheels = 6·5 ft.

Speed, 75 *miles per hour.*

Then $\quad \omega = \dfrac{75 \times 5280}{3600 \times 32\cdot5} = 33\cdot85$ radians per sec.

$$\frac{\omega^2}{g} = 35\cdot6 \text{ ft.}^{-1}$$

$$\mathrm{M}r = 583\cdot2 \text{ lb.-ft.,} \qquad \sqrt{2}\mathrm{M}r = 824\cdot6 \text{ lb.-ft.}$$

For inside-cylinders,

$$\mathrm{M}rl = 1119{\cdot}7 \text{ lb.-ft.}^2, \qquad \frac{1}{\sqrt{2}}\mathrm{M}rl = 791{\cdot}6 \text{ lb.-ft.}^2$$

$$\sqrt{\tfrac{1}{2}\left\{1+\left(\tfrac{l}{\mathrm{L}}\right)^2\right\}} = 0{\cdot}757, \quad \mathrm{M}r\sqrt{\tfrac{1}{2}\left\{1+\left(\tfrac{l}{\mathrm{L}}\right)^2\right\}} = 441{\cdot}5 \text{ lb.-ft.}$$

For outside cylinders,

$$\mathrm{M}rl = 3569 \text{ lb.-ft.}^2, \qquad \frac{1}{\sqrt{2}}\mathrm{M}rl = 2523 \text{ lb.-ft.}^2$$

$$\sqrt{\tfrac{1}{2}\left\{1+\left(\tfrac{l}{\mathrm{L}}\right)^2\right\}} = 1{\cdot}126, \quad \mathrm{M}r\sqrt{\tfrac{1}{2}\left\{1+\left(\tfrac{l}{\mathrm{L}}\right)^2\right\}} = 656{\cdot}7 \text{ lb.-ft.}$$

Taking $\frac{m}{\mathrm{M}} = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1, the results are exhibited in Tables III. and IV. for inside cylinders and outside cylinders respectively.

In coupled engines, the balance masses can be distributed between the driving-wheels, and the variation of vertical load on the rail distributed.

TABLE III.

Disturbing Forces and Couples in Two-cylinder Locomotives.

Inside Cylinders.

Proportion of masses balanced $= \frac{m}{\mathrm{M}}$	0		0·25		0·50		0·75		1·0	
	mr or mrl	Force or Couple.	mr or mrl	Force or Couple.	mr or mrl	Force or Couple.	mr or mrl	Force or Couple.	mr or mrl	Force or Couple.
Horizontal effort on train	824·6 lb.-ft.	13·11 ton	618·4 lb.-ft.	9·84 ton	412·3 lb.-ft.	6·55 ton	206·1 lb.-ft.	3·28 ton	0	0
Swaying couple	791·6 lb.-ft.^{2}	12·59 ton-ft.	593·7 lb.-ft.^{2}	9·45 ton-ft.	395·8 lb.-ft.^{2}	6·30 ton-ft.	197·9 lb.-ft.^{2}	3·15 ton-ft.	0	0
Vertical pressure on both rails	0	0	206·1 lb.-ft.	3·28 nto	412·3 lb.-ft.	6·55 ton	618·4 lb.-ft.	9·84 ton	824·6 lb.-ft.	13·11 ton
Transverse rocking couple	0	0	197·9 lb.-ft.^{2}	3·15 ton-ft.	395·8 lb.-ft.^{2}	6·30 ton-ft.	593 7 lb.-ft.^{2}	9·45 ton-ft.	791·6 lb.-ft.^{2}	12·59 ton ft.
Vertical pressure on each rail (hammer blow)	0	0	110·4 lb.-ft.	1·63 ton	220·8 lb.-ft.	3·27 ton	331·2 lb.-ft.	4·90 ton	441·5 lb.-ft.	6·54 ton

TABLE IV.

Disturbing Forces and Couples in Two-Cylinder Locomotives.

Outside Cylinders.

Proportion of masses balanced. $= \frac{m}{M}$	0		0·25		0·50		0·75		1·0	
	mr or *mrl*	Force or Couple.	*mr* or *mrl*	Force or Couple.	*mr* or *mrl*	Force or Couple.	*mr* or *mrl*	Force or Couple.	*mr* or *mrl*	Force or Couple.
Horizontal effort on train	824·6 lb.-ft.	13·11 ton	618·4 lb.-ft.	9·84 ton	412·3 lb.-ft.	6·55 ton	206·1 lb.-ft.	3·28 ton	0	0
Swaying couple	2523 lb.-ft.^{2}	40·11 ton-ft.	1893 lb.-ft.^{2}	30·09 ton-ft.	1261 lb.-ft.^{2}	20·05 ton-ft.	631 lb.-ft.^{2}	10·03 ton-ft.	0	0
Vertical pressure on both rails	0	0	206·1 lb.-ft.	3·28 ton	412·3 lb.-ft.	6·55 ton	618·4 lb.-ft.	9·84 ton	824·6 lb.-ft.	13·11 ton
Transverse rocking couple	0	0	631 lb.-ft.^{2}	10·03 ton-ft.	1261 lb.-ft.^{2}	20·06 ton-ft.	1893 lb.-ft.^{2}	30·09 ton-ft.	2523 lb.-ft.^{2}	40·11 ton-ft.
Vertical pressure on each rail (hammer blow)	0	0	164·2 lb.-ft.	2·61 ton	328·4 lb.-ft.	5·22 ton	492·6 lb.-ft.	7·83 ton	656·7 lb.-ft.	10·44 ton

CHAPTER VI

INERTIA FORCES OF SECOND AND HIGHER ORDERS

(1) **Velocity of Piston.**—Let OP be the position of the crank at any instant, PC the connecting-rod, OC the line of stroke (Fig. 1). It can be shown that the motion at any instant of a rigid body in a plane can be expressed as a motion of rotation about an axis at right angles to the plane of

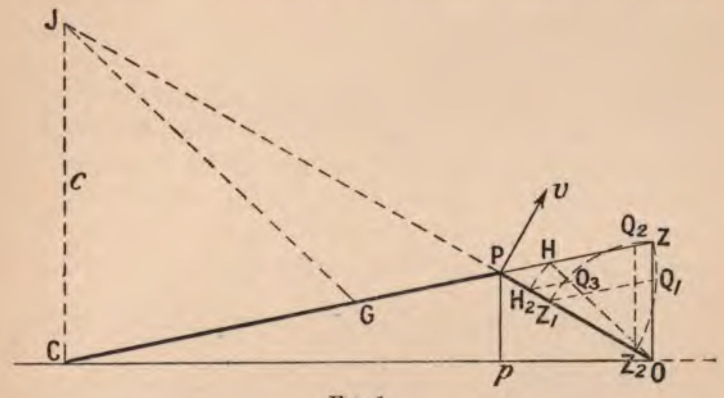

FIG. 1.

the motion. This axis is called the *instantaneous axis* of rotation, and the point of intersection of the instantaneous axis with the plane is called the *instantaneous centre.* The connecting-rod CP being considered rigid, the two points C and P are, at the instant under consideration, each rotating about the *instantaneous centre* of rotation of the connecting-rod. But the motion at P is, at the instant, in a direction at

right angles to OP; the point P may therefore be considered to rotate about any point in OP or OP produced. For all circles drawn through P, and having their centres on OP or OP produced, will have the same tangent at P. Similarly, the point C moving at the instant in the direction CO may be considered to rotate about any point on the line C*c*, drawn through C at right angles to CO. Thus, if both P and C are at the instant rotating about the same point, that point must be J, the point of intersection of OP and C*c*. J is therefore the *instantaneous centre* of rotation of the connecting-rod. Of course it must be clearly understood that the instantaneous centre changes its position from instant to instant as the crank revolves.

Let v and V respectively be the linear velocities of the crank-pin P and cross-head C, at the instant. Then, since P and C are points on the same rigid body rotating about J, their linear velocities are proportional to their radii from the centre of rotation. That is

$$\frac{V}{v} = \frac{JC}{JP} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In some positions of the crank the point J will move off the paper, so the relation (1) cannot always be used. Draw OZ at right angles to the line of stroke, and let the connecting-rod, produced if necessary, cut OZ at Z, then evidently the triangles JPC and OPZ are similar, and therefore their corresponding sides are proportional; therefore

$$\frac{JC}{JP} = \frac{OZ}{OP}, \text{ and } \frac{V}{v} = \frac{OZ}{OP}, \text{ or } V = \frac{v}{OP} \cdot OZ \,. \quad (2)$$

Therefore, from (2), if v the speed of the crank-pin is constant, since OP is of constant length, the velocity of the cross-head is proportional to the intercept OZ.

(2) **True Acceleration of Cross-head—Klein's Graphical Method.**—Let OP and PC be the crank and connecting-rod in any position (Fig. 2), the cross-head C reciprocating along the line OC. From O draw OZ at right angles to CO, to cut the axis of the connecting-rod (produced if necessary) at Z. With centre P and radius PZ draw a circle, and on

CP as diameter draw a circle. Draw the common chord AB of the two circles, and let it cut OC (produced if necessary)

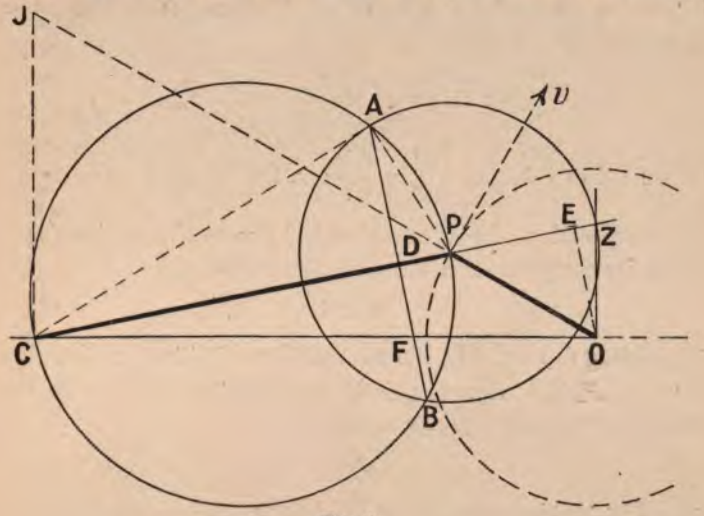

FIG. 2.

at F. Then $\hat{\mathrm{FO}}$ represents the acceleration of the cross-head C to the same scale that $\hat{\mathrm{PO}}$ represents the radial acceleration $\omega^2 r$ of the crank-pin P.

Proof.—The motion of the connecting-rod at any instant may be resolved into two component motions: (*a*) a motion of translation, like that of the coupling-rod of a locomotive, the velocities of all points on the rod being the same as that of the crank-pin; (*b*) a motion of oscillation about the crank-pin P as centre.

Let AB cut CP at D. Draw OE perpendicular to CP, meeting it (produced if necessary) at E. Find the instantaneous centre of rotation J of the connecting-rod, by drawing CJ at right angles to OC, and producing OP to cut CJ at J. Join PA and AC.

If v is the linear speed of the crank-pin P, the angular speed ω of the crank is $v/\overline{\mathrm{OP}}$, the angular speed ω_c of the connecting-rod is $v/\overline{\mathrm{JP}}$. Thus

$$\frac{\omega_c}{\omega} = \frac{\mathrm{OP}}{\mathrm{JP}} = \frac{\mathrm{ZP}}{\mathrm{CP}}, \quad i.e.\ \omega_c = \frac{\mathrm{ZP}}{\mathrm{CP}}\omega.$$

The acceleration of the cross-head C due to the component motion (*b*) is made up of a radial component in the direction CP, and a tangential component at right angles to CP. The magnitude of the former is

$$\omega_c^2 \times \mathrm{CP} = \omega^2 \times \frac{\overline{\mathrm{ZP}}^2}{\mathrm{CP}} = \omega^2 \cdot \overline{\mathrm{DP}} \quad . \quad . \quad (1)$$

$$\left(\text{since PZ} = \text{PA, and } \frac{\mathrm{DP}}{\mathrm{AP}} = \frac{\mathrm{AP}}{\mathrm{CP}};\ \therefore \mathrm{DP} = \frac{\mathrm{ZP}^2}{\mathrm{CP}}\right)$$

The acceleration of the cross-head C due to the motion (*a*) is ω^2PO, its component in the direction CP is

$$\omega^2\mathrm{PE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2), the acceleration of C in the direction CP is $\omega^2\overline{\mathrm{DE}}$. But the total acceleration of C is in the direction CO, and must be equal to

$$\omega^2\overline{\mathrm{FO}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

since $\overline{\mathrm{DE}}$ is the projection of $\overline{\mathrm{FO}}$ on CP.

If the construction of Fig. 2 be repeated for various positions of the crank, the various values of OF may be

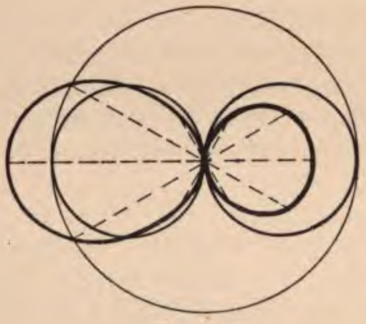

Fig. 3.

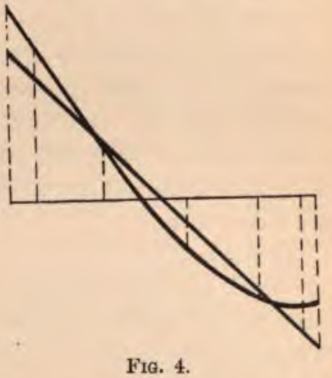

Fig. 4.

plotted to give a curve of cross-head acceleration. If the values of OF are set off along the corresponding crank position, a *polar* curve (Fig. 3) is obtained; if set off as ordinates at the cross-head C, a rectangular curve with *space* base (Fig. 4). If along a straight base distances be marked off proportional to

the angles the crank has described from the line of stroke OC, and the various values of OF are plotted as ordinates, a rectangular curve with *time* or crank-angle base (Fig. 5) is obtained, presuming the angular speed of the crank-shaft is uniform.

Figs. 3–5 are drawn for a connecting-rod length 3·5 times the crank length, $q = r/l = 1/3{\cdot}5$, but Fig. 2 is drawn for a

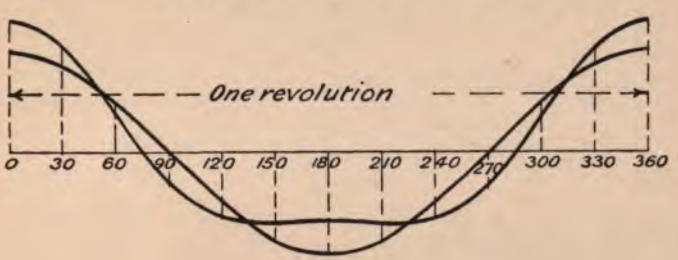

FIG. 5.

much shorter connecting-rod, so as to show clearly the construction. The two circles in Fig. 3, the straight line in Fig. 4, and the sine curve in Fig. 5, correspond to an infinitely long connecting-rod.

When the crank is on either dead centre,

$$ZP = OP, \text{ and } DP = r^2/l = qr.$$

On the inner dead centre, therefore,

$$FO = (r + qr) = (1 + q)r \quad . \quad . \quad . \quad . \quad (4)$$

on the outer dead centre,

$$FO = (1 - q)r \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The difference of the ordinates to the two curves (Fig. 4 or 5) is the acceleration due to the *obliquity* of the connecting-rod.

By multiplying the ordinates of the Klein acceleration curve (Fig. 5) by the reciprocating mass m, the values of the acceleration forces are obtained.

With two or more cylinders, the resultant acceleration force can be shown for different angles of revolution of the

shaft, by drawing Klein curves of accelerating force for each cylinder, displaced to correspond with the angles between the cranks, and taking the algebraic sum of the ordinates at each value of the abscissa. The resultant ordinate is the resultant unbalanced force for that position of the crank-shaft. Figs.

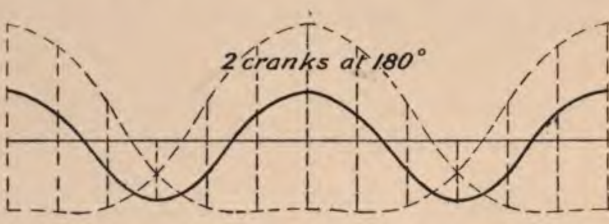

Fig. 6.

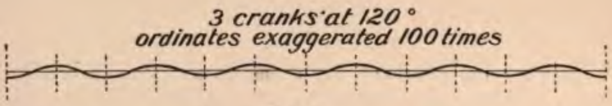

Fig. 7.

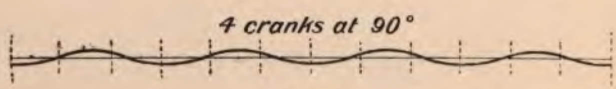

Fig. 8.

6, 7, and 8 show the resultant unbalanced force in engines respectively with two cranks at 180°, three cranks at 120°, and four cranks at 90°; the reciprocating masses being the same for each cylinder, likewise the ratio of connecting-rod length to crank radius. If these are different for the different cylinders the component Klein curves will be different, but the method, otherwise, is the same. The graphical method with its cumulative errors is not capable of sufficient accuracy to show clearly the shapes of the Klein curves for three and four equal cranks. Figs. 7 and 8 have therefore been calculated from sect. 3; the ordinates of Fig. 7 are exaggerated 100 times.

(3) **Acceleration of Cross-head—Analytical Method.**—Let

r = length of crank.
l = length of connecting-rod.
$q = r/l$ = ratio of crank to connecting-rod length.
θ = angle the crank has revolved from the inner dead centre.
ϕ = angle connecting-rod makes with line of stroke.
ω = angular speed of crank-shaft.

The exact expression for the acceleration of the cross-head or piston, which is investigated below, is

$$a = -\omega^2 r(\cos\theta + B_2\cos 2\theta + B_4\cos 4\theta + B_6\cos 6\theta + \ldots) \quad (1)$$

The exact values of the coefficients B_2, B_4, are given below; but taking their approximate values (1) may be written

$$a = -\omega^2 r(\cos\theta + q\cos 2\theta - \tfrac{1}{4}q^3\cos 4\theta + \tfrac{9}{128}q^5\cos 6\theta - \ldots) \quad (2)$$

Multiplying both sides of (1) by m, the mass reciprocating with the cross-head, we get the value of the accelerating force at the instant. If expressed in gravitation units, we have

$$F = \frac{ma}{g} = \frac{mr\omega^2}{g}(\cos\theta + B_2\cos 2\theta + B_4\cos 4\theta + B_6\cos 6\theta + \ldots) \quad (3)$$

Analytical Investigation.

In Fig. 9, let $x = OC = r\cos\theta + l\cos\phi$ . . . . . (1)

Then, $pP = r\sin\theta = l\sin\phi$;

$\therefore \sin\phi = q\sin\theta$, and

$\cos\phi = \sqrt{1 - q^2\sin^2\theta}$ . . . . . (2)

$\cos\phi$ is evidently a periodic function of θ, and can therefore be expanded by means of Fourier's Theorem. The values of $\cos\phi$ are the same for two crank positions at equal angles on opposite sides of OC, *i.e.* when $\theta = \theta_1$ and $-\theta_1$ respectively. Fourier's expansion must therefore contain only

cosines of multiples of θ; since $\cos\theta_1 = \cos(-\theta_1)$. Further, the values of $\cos\phi$ are the same for $\theta = \theta_1$ and $\theta = (\theta_1 + \pi)$. That is, the period of $\cos\phi$ corresponds to a difference π in the value of θ, while the period of $\cos\theta$ corresponds to a

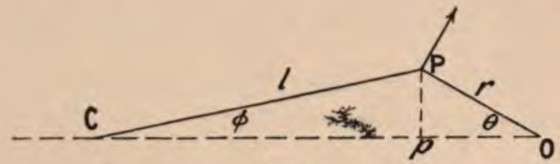

FIG. 9.

difference 2π. The expansion for $\cos\phi$ must therefore contain only cosines of multiples of 2θ.

However, it is more convenient to proceed by another method. Expanding the expression on the right-hand side of (2) by the Binomial Theorem

$$\cos\phi = 1 - \tfrac{1}{2}q^2\sin^2\theta - \tfrac{1}{8}q^4\sin^4\theta - \tfrac{1}{16}q^6\sin^6\theta \quad (3)$$

The expression for $\cos\phi$ in (3) can be transformed into an expansion in terms of cosines of multiples of 2θ by means of the formula (see Todhunter's "Trigonometry")

$$2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - n\cos(n-2)\theta + \frac{n(n-1)}{1\cdot2}\cos(n-4)\theta + \dots + (-1)^{\frac{n}{2}}\frac{n\cdot(n-1)\dots(\frac{1}{2}n+1)}{2\cdot1\cdot2\dots\frac{n}{2}}. \quad (4)$$

When $n = 2$, (4) becomes

$$-2\sin^2\theta = \cos 2\theta - \frac{2}{2\cdot1}$$

$$i.e.\ \sin^2\theta = \tfrac{1}{2} - \frac{\cos 2\theta}{2}$$

When $n = 4$, (4) becomes

$$2^3\sin^4\theta = \cos 4\theta - 4\cos 2\theta + \frac{4\cdot3}{2\cdot1\cdot2}$$

$$i.e.\ \sin 4\theta = \tfrac{3}{8} - \tfrac{1}{2}\cos 2\theta + \tfrac{1}{8}\cos 4\theta$$

When $n = 6$, (4) becomes

$$-2^5 \sin^6\theta = \cos 6\theta - 6 \cos 4\theta + \frac{6 \cdot 5}{1 \cdot 2} \cos 2\theta - \frac{6 \cdot 5 \cdot 4}{2 \cdot 1 \cdot 2 \cdot 3}$$

$$i.e.\ \sin^6\theta = \tfrac{5}{16} - \tfrac{15}{32} \cos 2\theta + \tfrac{3}{16} \cos 4\theta - \tfrac{1}{32} \cos 6\theta$$

Similarly, the expressions for the sines of the higher multiples of θ, may be obtained.

Therefore, making the above substitutions for $\sin^2\theta$, $\sin^4\theta$, . . . , (3) may be written

$$\cos \phi = A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta + A_6 \cos 6\theta + \dots \quad (5)$$

where

$$\begin{aligned} A_0 &= 1 - \tfrac{1}{4}q^2 - \tfrac{3}{64}q^4 - \tfrac{5}{256}q^6 - \dots \\ A_2 &= \phantom{1 -{}} \tfrac{1}{4}q^2 + \tfrac{1}{16}q^4 + \tfrac{15}{512}q^6 + \dots \\ A_4 &= \phantom{1 - \tfrac{1}{4}q^2} - \tfrac{1}{64}q^4 - \tfrac{3}{256}q^6 - \dots \\ A_6 &= \phantom{1 - \tfrac{1}{4}q^2 - \tfrac{1}{64}q^4 -{}} \tfrac{1}{512}q^6 + \dots \end{aligned}$$

a, the acceleration of A, is equal to $\dfrac{d^2x}{dt^2}$

Differentiating twice the expression (1)

$$x = r \cos \theta + l(A_0 + A_2 \cos 2\theta + A_4 \cos 4\theta + \dots)$$

we get

$$a = -\omega^2 r \cos \theta - \omega^2 l(4A_2 \cos 2\theta + 16A_4 \cos 4\theta + 36A_6 \dots)$$

Or, remembering that $l = r/q$,

$$\begin{aligned} a = -\omega^2 r(&\cos \theta + B_2 \cos 2\theta + B_4 \cos 4\theta \\ &+ B_6 \cos 6\theta + \dots) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6) \end{aligned}$$

where

$$\begin{aligned} B_2 &= 4A_2/q &&= q + \tfrac{1}{4}q^3 + \tfrac{15}{128}q^5 + \dots \\ B_4 &= 16A_4/q &&= - \tfrac{1}{4}q^3 - \tfrac{3}{16}q^5 - \dots \\ B_6 &= 36A_6/q &&= \phantom{q - \tfrac{1}{4}q^3 -{}} \tfrac{9}{128}q^5 + \dots \end{aligned}$$

In approximations, the first term only of these expressions may be used, and (6) may then be written

$$\begin{aligned} a = -\omega^2 r(&\cos \theta + q \cos 2\theta - \tfrac{1}{4}q^3 \cos 4\theta \\ &+ \tfrac{9}{128}q^5 \cos 6\theta - \dots) \quad . \quad . \quad . \quad . \quad (7) \end{aligned}$$

Table V. gives the accurate values of B_2, B_4, B_6 for various values of q.

TABLE V.

COEFFICIENTS FOR INERTIA FORCES OF SECOND, FOURTH, AND SIXTH ORDERS.

$\frac{l}{r}=\frac{1}{q}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6	7
B_2	0·2918	0·2540	0·2250	0·2020	0·1678	0·1436
B_4	−0·0062	−0·0041	−0·0028	−0·0020	−0·0012	−0·0007
B_6	0·0001	0·0001	0·0000	0·0000	0·0000	0·0000

(4) **Simple Harmonic Motions from Imaginary Cranks revolving at 2ω, 4ω, 6ω.**—The first term on the right-hand side of equation (3), sect. 3, is the primary force of acceleration. The second term $\frac{-mr\omega^2}{g}q\cos 2\theta$ is the *secondary* force of acceleration. Its value is greatest when $\cos 2\theta = \pm 1$, *i.e.* when $\theta = 0$; 90°, 180°, 270° . . . , and is then $\pm mrq\omega^2/g$. The maximum value of the secondary force is therefore equal to that of the primary force of acceleration multiplied by the ratio of crank to connecting-rod length. Its value at any instant is the projection on the line of stroke of a centrally directed force rotating at a speed 2ω, twice that of the crank. It may be considered as due to the simple harmonic motion of a mass driven from an imaginary crank of the same radius as the actual crank, but revolving at twice the speed.

Similarly, the other terms in (3) may be considered as due to masses driven with simple harmonic motion from imaginary cranks revolving at 4, 6, 8 . . . times the speed of the engine shaft. They may be referred to as the inertia forces of the 4th, 6th, 8th, . . . orders respectively.

Since
$$\omega^2 B_2 = \frac{B_2}{4}(2\omega)^2$$
$$\omega^2 B_4 = \frac{B_4}{16}(4\omega)^2$$
$$\omega^2 B_6 = \frac{B_6}{36}(6\omega)^2, \text{ etc.,}$$

the masses attached to the imaginary cranks of the 2nd, 4th, 6th, . . . orders are $B_2/4$, $B_4/16$, $B_6/36$, . . . times respectively that of the primary mass. If $q = \frac{1}{5}$ (*i.e.* if the connecting-rod length is $2\frac{1}{2}$ times the stroke), these imaginary masses of the 2nd, 4th, 6th orders are $\frac{1}{20}$th, $\frac{1}{8000}$th, $\frac{1}{1,600,000}$th parts respectively of the primary mass. The corresponding inertia forces are $\frac{1}{5}$th, $\frac{1}{500}$th, $\frac{9}{400,000}$th parts respectively of the primary forces.

In an engine where the primary inertia forces are unbalanced, those of higher orders need not be considered. But if the primary inertia forces are balanced, the secondary inertia forces, being a comparatively large fraction of the primary, may still be objectionable if left unbalanced. For example, in a four-cylinder engine for a motor-car, the primary forces are balanced; the vibration produced when the engine is running while the car is at rest is due to the secondary forces which are unbalanced. When both primary and secondary forces are balanced, the residual inertia forces, those of the fourth and higher orders, should scarcely be appreciable.

(5) **Engine with Cylinders in the Same Axial Plane.**—Let Fig. 10 show the distances between the cylinders, and Fig. 11 the angles the cranks have described from OV, the

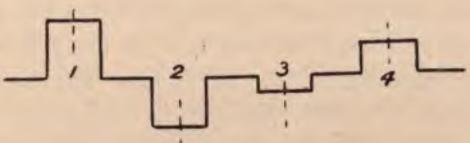

FIG. 10.

axial plane of the cylinders. Fig. 12 shows the corresponding positions of the imaginary cranks of the second order, Fig. 13 those of the fourth order; the angles in Figs. 12 and 13 being two and four times respectively those in Fig. 11. The resultant inertia force and couple of the second order are found by applying the methods of Chap. IV. to the cranks of Fig. 12; those of the fourth order by the same methods

applied to the cranks of Fig. 13; in each case the planes of reciprocation being the same as in Fig. 10.

Fig. 11.

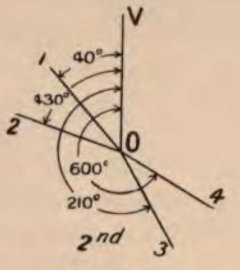

Fig. 12.

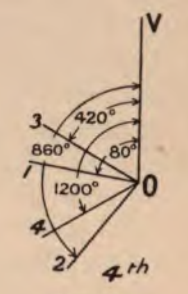

Fig. 13.

(6) **Conditions to be satisfied for Balance of Second and Higher Orders.**—The conditions for secondary balance can be stated as in sect. 20, Chap. III., provided the various crank angles are taken twice those of the actual cranks. That is, the secondary *mr* and *mrl* vector-polygons must be closed.

The conditions that the resultant force and resultant couple of any higher order should each be zero, may be similarly stated; *i.e.* the *mr* and *mrl* vector-polygons of that order must be closed.

If the engine is perfectly balanced for primary and higher orders of the inertia forces and couples, all the *mr* and *mrl* polygons must be closed. The *mr* polygons of the second, fourth, sixth, . . . , orders have their sides of the same length as the corresponding sides of the primary, but drawn at twice, four, six, . . . , times the corresponding angles to the reference line.

Each *mrl* vector-polygon has its sides parallel to the corresponding *mr* vector-polygon. The corresponding sides of the series of *mr* vector-polygons are all equal; likewise for the series of *mrl* vector-polygons. By taking the plane of transference at the axis of one cylinder, one side of each *mrl* polygon is zero.

Consider a primary and secondary closed vector-polygon of three sides, *i.e.* a triangle. Let ABC (Fig. 14) be the

primary vector-triangle, AB coinciding with the reference line OV. The secondary triangle ABC″ has then one side coincident with AB, and, of course, BC = BC″ and CA = C″A. Therefore the angle ABC = angle ABC″, and the triangles ABC and ABC″ are symmetrical about AB. But the angle which the vector BC″ has described from BV is twice the angle VBC, which vector BC has described. Therefore the angles, VBC, CBC″, C″BV are equal, and must therefore each be 120°; and ABC is 60°. Similarly, it can be shown that the angle CAB must be 60°. That is, the closed primary and secondary vector-triangles must be *equilateral*; similarly, that of the fourth order is equilateral.

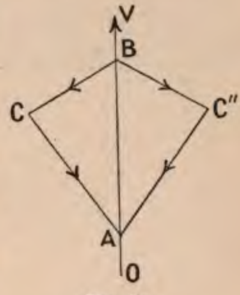

FIG. 14.

The vector diagrams of the sixth and twelfth order have their three sides coincident in direction, and therefore the force or couple of these orders is not zero.

Similarly, if a closed primary vector-polygon of n sides is equilateral and equiangular; then, if n is even, the sides of the vector-diagram of the nth order will coincide in direction; if n is odd, those of the $2n$th order. The vector-polygons of all lower orders will be closed. If the primary vector-polygon is not both equilateral and equiangular, some of the vector-polygons of lower order will be open. Thus it appears that no engine of the usual design (see sect. 2, Chap. V.) with cylinders all in the same axial plane can be mathematically in perfect balance.

If a vector-polygon is unclosed, it can be closed by adding another vector. A vector involves direction and length, a vector quantity concerned in the problem of balancing involves direction and either an mr or an mrl. Thus the condition that one vector-polygon is closed is equivalent to specifying two relations between the variables.

(7) **Perfect Primary and Secondary Balance.**—If the primary and secondary forces and couples vanish, this is equivalent to specifying eight relations between the

variables. With n cylinders involving $(3n - 4)$ variables, therefore, $(3n - 12)$ variables may presumably be assigned arbitrarily, leaving eight variables to be determined so as to satisfy the eight conditional equations.

(8) **Four-crank Engine with Perfect Primary and Secondary Balance.**—When $n = 4$, $(3n - 12) = 0$. Therefore, no other variables besides one value of mr and one distance between two planes of revolution can be assigned.

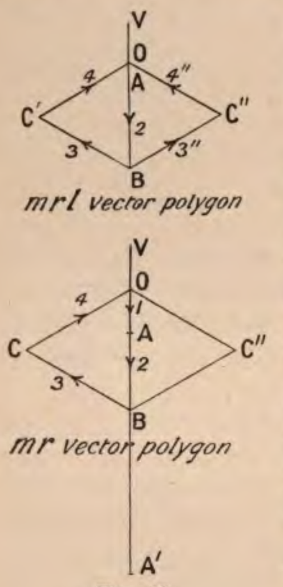

Fig. 15.

Suppose the first and second mrl vector-polygons drawn (Fig. 15) with respect to the plane of one of the cylinders, say No. 1; they therefore reduce to equilateral triangles (sect. 6), the side corresponding to m_1 being zero. Let the direction of one crank, No. 2, coincide with the axial plane of the cylinders OV; the secondary crank, No. 2, will also lie in the axial plane of the cylinders, and the sides $m_2r_2l_2$ of the primary and secondary mrl vector-polygons coincide, as at AB.

Drawing the two mr vector-polygons OABCO and OABC″O (Fig. 15), the sides 2, 3, 4 are parallel to those of the mrl polygon; and since both polygons must be closed, the side 1 must coincide with either of the sides 2, 3, 4; say No. 2, giving each four-sided polygon the form of an equilateral triangle.

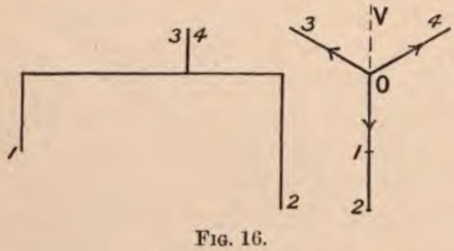

Fig. 16.

The cranks 1 and 2 must be in the same axial plane, but

may point in the same or opposite directions according as A lies between or outside O and B. The two remaining sides, 3 and 4, are therefore equal, *i.e.* $m_3r_3 = m_4r_4$. And since $m_3r_3l_3 = m_4r_4l_4$, l_3 must be equal to l_4. That is, the cranks 3 and 4 must be in the same plane of revolution.

Fig. 16 shows the relative position of the planes of revolution and the crank angles, when 1 and 2 are on opposite sides of the plane of cranks 3 and 4; Fig. 17 when on the same side. OABCO and OA'BCO (Fig. 15) are the *mr* vector-polygons corresponding to Figs. 16 and 17 respectively.

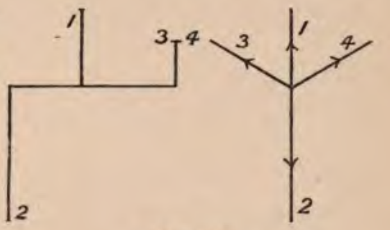

FIG. 17.

Summing up, the only theoretically possible arrangement of a four-crank engine with perfect primary and secondary balance is as follows, assuming the crank-radii all equal.

One pair of the cranks must (*a*) lie in the same plane of revolution, (*b*) at an angle of 120° with each other, and (*c*) have equal reciprocating masses. The other pair must (*d*) lie in the axial plane bisecting the angle between the pair at 120°; (*e*) if they point in the $\left\{\begin{matrix}\text{same}\\\text{opposite}\end{matrix}\right\}$ direction the $\left\{\begin{matrix}\text{sum}\\\text{difference}\end{matrix}\right\}$ of their reciprocating masses is equal to one of the masses of the equal pair, the direction of the crank having the greater mass being at 120° with each of the two equal mass cranks; (*f*) the distances of their planes of revolution from that of the pair at 120° must be inversely proportional to their masses.

Although the four-crank engine balanced as to primary and secondary inertia forces is not a practicable form, from it may be derived practicable forms of balanced five-crank and six-crank engines.

(9) **Five-crank Engine with Perfect Primary and Secondary Balance.**—Fig. 18 shows a practicable arrangement of a five-crank engine with perfect primary and

secondary balance, derived from a combination of two four-crank engines arranged as in Fig. 16. The planes of revolution of the cranks being numbered 1, 2, 3, 4, 5 from left to right, draw the crank-angle diagram with three lines OA, OB, OC, mutually at 120°. One four-crank set, as in Fig. 16, has its two coplanar equal masses m placed in

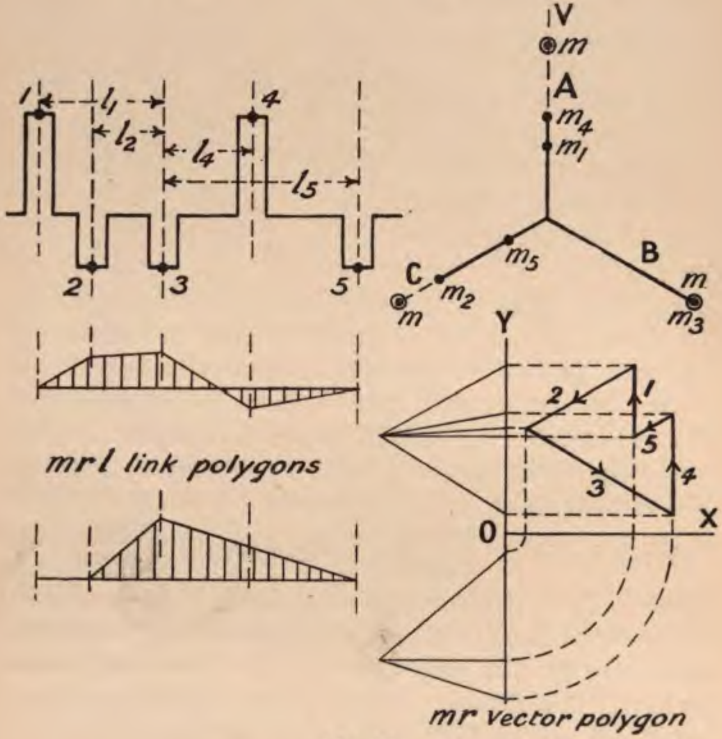

Fig. 18.

the plane 3, in the crank directions OB and OC; the two masses m_1 and m_4 in the plane 1 and 4 respectively, in the crank direction OA. The second four-crank set as in Fig. 16, has its two coplanar masses equal to those of the first set, and also placed in the plane 3, but in the crank directions OA and OB respectively; the two masses m_2 and m_5 in the planes 2 and 5 respectively in the crank direction OC. We

have thus an engine with perfect primary and secondary balance, having four cranks in separate planes of revolution 1, 2, 4, 5; and four cranks in one plane of revolution 3, two of them being coincident at OB. But an engine with three coplanar cranks at OA, OB and OC driving equal reciprocating masses would be in perfect primary and secondary balance. The three equal masses at A, B, and C may therefore be omitted from the system, leaving one crank in the plane 3, in direction OB.

If the cranks are all of the same radius, $m_3 = m_1 + m_4 = m_2 + m_5$, and $m_1l_1 = -m_4l_4$, $m_2l_2 = -m_5l_5$, the l's being measured from the plane 3.

As a verification of the balance, the primary mr vector-polygon and mrl link-polygons are shown. The secondary mr polygon is an image of the primary; the mrl secondary link-polygon in plane OY is identical with the primary, that in plane OX is an image of the primary.

The number of variables in a five-crank engine, which may be assigned arbitrarily with the condition of perfect primary and secondary balance, is $(3 \times 5 - 12) = 3$. If the centre line of the cylinders be given, defining three ratios between the l's, the solution is the arrangement given above. We cannot assign arbitrarily all the mr's or all the crank angles, each of which is equivalent to assigning four variables; and also impose the conditions for primary and secondary balance.

(10) **Engine with Five Cranks at 72°.**—Take the five cranks at equal angles 72° (Fig. 19), and all the mr's equal. Thus, eight variables are fixed, but in choosing them we have assured that the primary and secondary forces are balanced; so, in fact, are those of the fourth, sixth, and eighth orders. Numbering the planes of revolution in order from left to right, and the cranks as indicated by the inner ring of figures in the crank-angle diagram, the primary mr vector-polygon is a regular pentagon. The crank-angle diagrams of the second, fourth, . . . orders will show five cranks at angles of 72°, but the sequence of numbers will be as shown in the corresponding rings of figures. The mr vector-polygons

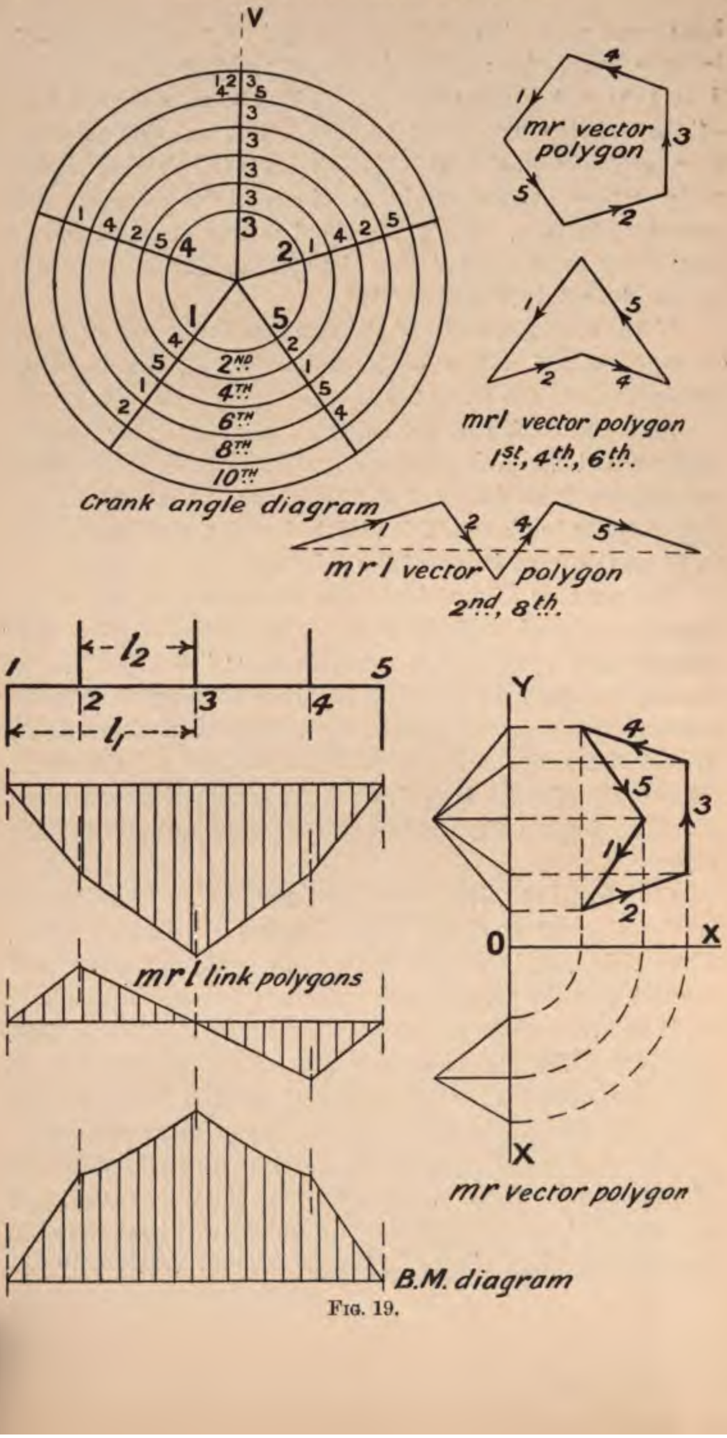
V
mr vector polygon
Crank angle diagram
2ND
4TH
6TH
8TH
10TH
mrl vector polygon
1st, 4th, 6th.
mrl vector polygon
2nd, 8th.
l_2
l_1
Y
X
O
X
mrl link polygons
mr vector polygon
B.M. diagram

Fig. 19.

are all coincident with the regular pentagon drawn, but the sides are numbered differently.

There are still three variables to be assigned; we may determine two from the condition that the primary couples are to be balanced, leaving still one relation between the variables which may be assigned. Assume that $l_1 = -l_5$, measuring l_1 and l_3 from the middle crank 3.

Draw the primary *mrl* vector-polygon with respect to plane 3; its sides are parallel to those of the *mr* vector-polygon, but its side 3 is of zero length. Since $l_1 = -l_5$, the sides 1 and 5 are equal; thus determining that the sides 2 and 4 must also be equal, and $l_2 = -l_4$. The engine is therefore symmetrical about the plane 3. The distances l_1 and l_2 are proportional to the sides 1 and 2 of the primary *mrl* vector-polygon.

Therefore

$$\frac{l_1}{l_2} = \frac{\cos 18^\circ}{\cos 54^\circ} = \frac{\sqrt{5}+1}{2} = 1{\cdot}618$$

Thus all the variables are determined.

The secondary *mrl* vector-polygon is unclosed, showing a secondary unbalanced couple. The value of the resultant *mrl* is

$$\begin{aligned} &2mr(l_1 \cos 18^\circ + l_2 \cos 54^\circ) \\ &= 2mrl_1\left(\cos 18^\circ + \frac{\cos^2 54^\circ}{\cos 18^\circ}\right) \\ &= 2{\cdot}662 mrl_1 \end{aligned}$$

The maximum value of the resultant secondary couple is therefore $2{\cdot}662\ mrl_1 q\omega^2/g$.

If $m = 4$ lbs., $r = 2$ in. $= \frac{1}{6}$ ft., $l_1 = 16$ in. $= 1\frac{1}{3}$ ft., $q = \frac{1}{5}$, $\omega = 104{\cdot}7$ radians per second corresponding to 1000 revs. per min., the maximum value of the couple is 134·3 foot-lbs. The *mrl* vector-polygons of the fourth and sixth orders are coincident with the primary, if the sequence of numbers be altered; that is, there are no unbalanced couples of the fourth and sixth orders.

The *mrl* link-polygons are shown, and from them the resultant bending-moment diagram is obtained.

If in a five-crank engine with cranks at 72°, and equal piston masses, the distances between consecutive cylinders are each equal to l, there is an unbalanced primary longitudinal couple. Fig. 20 shows the primary mrl vector-polygon relative to the plane of the crank No. 3. The length of the sides 1 and 5 is $2mrl$, that of the sides 2 and 4 is mrl. The resultant mrl vector is evidently

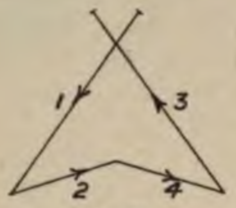

FIG. 20.

$$2mrl\,(2 \cos 54^\circ - \cos 18^\circ)$$
$$= 0{\cdot}449\, mrl.$$

The resultant primary couple is therefore

$$0{\cdot}449 mrl\omega^2/g.$$

The secondary mrl vector-polygon will be somewhat like that shown in Fig. 19, but the two long sides will be twice the length of the two shorter sides. The resultant mrl vector is then

$$2mrl\,(2 \cos 18^\circ + \cos 54^\circ)$$
$$= 4{\cdot}980\, mrl.$$

The maximum value of the resultant secondary couple is therefore $4{\cdot}980 mrlq\omega^2/g$, which, with usual values of q, will be more than twice the primary.

For motor-car engines, this engine deserves some attention from manufacturers. Its balance is better than that of the four-cylinder engine. With the four-stroke Otto cycle, the impulses from the various cylinders slightly overlap, so that the uniformity of torque is better than in the four-cylinder engine. In these two respects it is slightly inferior to the six-cylinder engine. In respect of constancy of kinetic energy of the pistons and connecting-rods, it is much superior to the six-cylinder engine. (See Chap. IX.)

Six-crank Engine.—By combining three sets of the four-crank engine (Fig. 16 or 17), a practicable six-crank engine is obtained. The three pairs of coplanar cranks

at 120° of the three sets are placed in the same plane LL (Fig. 21), so as to have the six equal masses grouped in three pairs at 120°. These evidently are self-balancing, and can therefore be omitted. Let the cranks 1 and 6 be the remaining pair of the first set. Taking Fig. 16 as the basis,

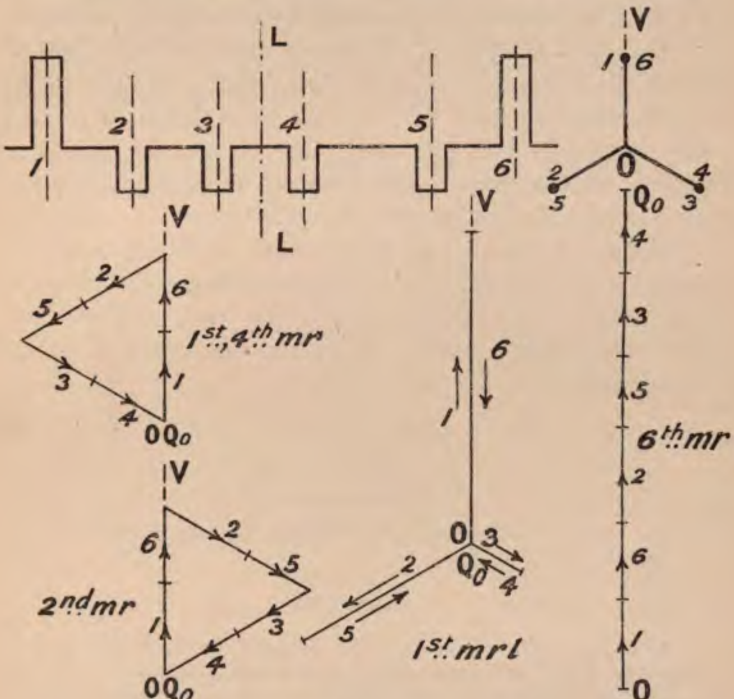

FIG. 21.

and measuring the l's from the plane LL, the cranks 1 and 6 point in the same direction. The cranks 2 and 5 of the second set make an angle of 120° with cranks 1 and 6, and cranks 3 and 4 of the third set are at angles of 120° mutually with 1, 6, 2, 5. Also

$$m_1r_1l_1 = -m_6r_6l_6 \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$m_2r_2l_2 = -m_5r_5l_5 \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$m_3r_3l_3 = -m_4r_4l_4 \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and $(m_1r_1 + m_6r_6) = (m_2r_2 + m_5r_5) = (m_3r_3 + m_4r_4)$. (4)

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Fig. 21 shows the arrangement of cranks, and the various vector-polygons. The first, second, and fourth mr vector-polygons are closed, the sixth mr has a resultant $OQ_0 = 3(m_1r_1 + m_6r_6)$. The first mrl vector-polygon with respect to the plane LL is shown; it consists of three pairs of lines starting from and returning to the origin. If the equations (1) to (3) are satisfied, all the mrl vector-polygons are closed.

If $l_1 = -l_6$, $l_2 = -l_5$, $l_3 = -l_4$ and the radii are all equal, the m's must necessarily be all equal, and the engine is symmetrical about LL; but the three distances l_1, l_2, l_3 may be assumed at pleasure. The six-cylinder motor-car engine has the crank angles arranged as in Fig. 21, usually with equal distances between the cylinders.

With a two-stroke cycle six-crank engine, it would be desirable for uniformity of torque to have the six cranks at 60°. Fig. 17 can then be used as the basis, and the three sets combined in a manner similar to that described above. The equations for this case are

$$\begin{aligned} m_1r_1l_1 &= m_6r_6l_6 \\ m_2r_2l_2 &= m_5r_5l_5 \\ m_3r_3l_3 &= m_6r_6l_6 \end{aligned}$$

$$(m_1r_1 - m_6r_6) = (m_2r_2 - m_5r_5) = (m_3r_3 - m_4r_4).$$

Exercises.

1. Explain what is meant by the instantaneous axis of a link. A horizontal engine has a connecting-rod five cranks in length. Find the velocity ratio of slide-block and crank-pin at one quarter stroke. What is the mean velocity ratio of crank-pin and slide-block? (Inst. C.E., Feb. 1903.)

Ans. $1 : 1{\cdot}09$; $\frac{\pi}{2} : 1$.

2. What is meant by the *instantaneous centre* of a force moving in a fixed plane? A rigid body has a plane motion. Three points on the body A, B, and C, in the same plane, are such that AB = 3 feet, BC = 2 feet, and AC = 2·6 feet. At a certain instant it is known that the point A has a velocity of 4 feet per second in the direction from A to C, and that the point B is moving in the direction from C to B. Show how the velocity of any other point on the body may be obtained, graphically or otherwise, and determine the values for the velocities of B and the point midway between A and B. (Inst. C.E., Feb. 1899.)

Ans. 5·8 ft./sec.; 3·2 ft./sec.

3. In a steam-engine the piston at the beginning of its stroke is exposed to a total pressure of 2000 lbs., but the inertia is such that the thrust of the piston-rod at the cross-head is only 1600 lbs. The speed of the engine is now raised until it becomes half as great again as before, while the steam pressure is unchanged: what is the thrust of the piston-rod? (Inst. C.E., Feb. 1899.) *Ans.* 1100 lbs.

4. If the figures in Example 3 refer to the piston at the beginning of its *forward* stroke (*i.e.* when the crank-pin is between the shaft and the cylinder), and the connecting-rod length is twice the stroke, find the thrust of the piston-rod at the cross-head when the piston is beginning its backward stroke, for both the initial and the increased speeds of the engine. *Ans.* 1760 lbs., 1460 lbs.

5. Discuss generally, the influence of "short" and "long" connecting-rods on the magnitude of the unbalanced force in a four-cylinder motor-car engine.

CHAPTER VII

TRANSVERSE COUPLES DUE TO CONNECTING-RODS

(1) **Angular Acceleration of Connecting-rod.**—The linear motion of the mass-centre of the connecting-rod having been accounted for by replacing its mass by two masses at the crank-pin and cross-head respectively (sect. 1, Chap. V.), there still remains a transverse couple required to give the rod its angular acceleration at any instant. The opposite couple of reaction tends to rock the engine frame about an axis parallel to that of the crank-shaft.

Using the notation of sect. 3, Chap. VI., the angular acceleration of the rod when the crank is at the angle θ with the dead centre is

$$\alpha = -\omega^2(C_1 \sin \theta + C_3 \sin 3\theta + C_5 \sin 5\theta + \ . \ . \ .) \quad (1)$$

The exact values of C_1, C_3, C_5, . . . are investigated below. It will usually be sufficiently accurate to take the first term in each series, in which case (1) may be written

$$\alpha = -\omega^2(q \sin \theta - \tfrac{3}{8}q^3 \sin 3\theta + \tfrac{15}{128}q^5 \sin 5\theta - \ . \ . \ .) \quad (2)$$

Multiplying both sides of (2) by I/g, where I is the moment of inertia of the rod about an axis through its mass-centre parallel to the crank-shaft, we get the value of the transverse couple of acceleration, in gravitation units.

$$C = -\frac{I\omega^2}{g}(C_1 \sin \theta + C_3 \sin 3\theta + C_5 \sin 5\,\theta + \ . \ . \ .) \quad (3)$$

(2) **Analytical Investigation.**—From Fig. 9, Chap. VI.

$$pP = r \sin \theta = l \sin \phi$$

$$\sin \phi = \frac{r}{l} \sin \theta = q \sin \theta$$

$$\text{Let } x = \sin \phi\,; \text{ then } \phi = \sin^{-1}x$$

Expanding by Maclaurin's theorem

$$\phi = x + \tfrac{1}{6}x^3 + \tfrac{3}{40}x^5 + \tfrac{15}{336}x^7 + \ldots$$

that is

$$\phi = q \sin \theta + \tfrac{1}{6}q^3 \sin^3\theta + \tfrac{3}{40}q^5 \sin^5\theta + \ldots \quad (1)$$

Using the general formula (see Todhunter's "Trigonometry")

$$2^{(n-1)}(-1)^{\frac{n-1}{2}} \sin^n\theta = \sin n\theta - n \sin (n-2)\theta + \frac{n(n-1)}{1 \cdot 2} \sin^{n-4}\theta$$

$$+(-1)^{\frac{n-1}{2}} \frac{n(n-1) \ldots \frac{1}{2}(n+3)}{\left\lfloor \frac{n-1}{2} \right.} \sin \theta \quad (2)$$

We get, when $n = 3$,

$$-2^2 \sin^3\theta = \sin 3\theta - 3 \sin \theta$$

or,

$$\sin^3\theta = -\tfrac{1}{4} \sin 3\theta + \tfrac{3}{4} \sin \theta \quad . \quad . \quad (3)$$

and when $n = 5$,

$$2^4 \sin^5\theta = \sin 5\theta - 5 \sin 3\theta + 10 \sin \theta$$

or,

$$\sin^5\theta = \tfrac{1}{16} \sin 5\theta - \tfrac{5}{16} \sin 3\theta + \tfrac{5}{8} \sin \theta \quad . \quad (4)$$

Substituting from (3) and (4) in (1), we have

$$\phi = A \sin \theta + A_3 \sin 3\theta + A_5 \sin 5\theta + \quad . \quad (5)$$

where A_1, A_3, . . . have the following values:—

$$\begin{aligned} A_1 &= q + \tfrac{1}{8}q^3 + \tfrac{3}{64}q^5 + \ldots \\ A_3 &= \quad - \tfrac{1}{24}q^3 - \tfrac{3}{128}q^5 - \ldots \\ A_5 &= \qquad\qquad\quad \tfrac{3}{640}q^5 + \ldots \end{aligned}$$

Taking for A_1, A_3, . . . the first term in each series, we have approximately

$$\phi = q \sin \theta - \tfrac{1}{24}q^3 \sin 3\theta + \tfrac{3}{640}q^5 \sin 5\theta \quad . \ (5a)$$

The angular velocity of the connecting-rod is

$$\frac{d\phi}{dt} = \frac{d\phi}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{d\phi}{d\theta},$$

$$\therefore \frac{d\phi}{dt} = \omega(A_1 \cos \theta + 3A_3 \cos 3\theta + \ldots) \quad . \quad (6)$$

Similarly, the angular acceleration a of the rod is

$$\frac{d^2\phi}{dt^2} = -\omega^2(A_1 \sin\theta + 9A_3 \sin 3\theta + 25A_5 \sin 5\theta + \ldots)$$

or, $$a = -\omega^2(C_1 \sin\theta + C_3 \sin 3\theta + C_5 \sin 5\theta + \ldots) \quad (7)$$

where
$$\begin{aligned} C_1 &= A_1 &&= q + \tfrac{1}{8}q^3 + \tfrac{3}{64}q^5 + \ldots \\ C_3 &= 9A_3 &&= \quad -\tfrac{3}{8}q^3 - \tfrac{27}{128}q^5 - \ldots \\ C_5 &= 25A_5 &&= \qquad\qquad \tfrac{15}{128}q^5 + \ldots \end{aligned}$$

Table VI. gives the accurate values of C_1, C_3, C_5, for various values of q.

TABLE VI.

COEFFICIENTS FOR TRANSVERSE COUPLES OF FIRST, THIRD, AND FIFTH ORDERS.

$\frac{l}{r} = \frac{1}{q}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6	7
C_1	0·2887	0·2520	0·2236	0·2010	0·1673	0·1432
C_3	−0·0091	−0·0060	−0·0042	−0·0030	−0·0017	−0·0011
C_5	0·0002	0·0001	0·0001	0·0000	0·0000	0·0000

(3) **Transverse Couples, Primary and of Higher Orders.**—The first term in the expression for C may be called the primary transverse couple, the successive terms the transverse couples of the third, fifth, . . . orders. The transverse couples of the third, fifth, . . . orders are evidently the same as primary transverse couples of engines running at three, five, . . . times the speed of the crank-shaft. If $q = \frac{1}{5}$, the maximum values of the transverse couples of the third and fifth orders are approximately $\frac{3}{200}$ths and $\frac{3}{16000}$ths that of the primary.

The maximum value of the primary transverse couple occurs when $\sin\theta = \pm 1$, *i.e.* when the crank is at right angles to the line of stroke, and is evidently $\mp IC_1\omega^2/g$. The primary transverse couple may therefore be regarded as the projection on a line at right angles to the line of stroke, of a

vector quantity of constant magnitude rotating with the crank.

(4) **Resultant Primary Transverse Couple.**—If the engine has several cranks on the crank-shaft, the resultant transverse couple is obtained by algebraic addition of the transverse couples of the several connecting-rods, since the axis of each transverse couple is parallel to the crank-shaft. If q is the same for all the rods, so also is C_1, C_3 . . . Thus drawing a vector-polygon with its sides parallel to the various cranks, and of lengths I_1, I_2 . . . , the resultant transverse couple at the instant is the projection on a line at right angles to the line of stroke of the resultant vector I multiplied by $C_1\omega^2/g$. If the connecting-rods are in balance as regards the primary transverse couple, the primary I vector-polygon must be closed.

If in the engines with cranks as represented in Figs. 6, 7, 11, Chap. III.; Fig. 1, Chap. V.; and Figs. 19, 21, Chap. VI.; the cranks and connecting-rod are all similar, the primary I vector-polygon is closed in each case, and the connecting-rods are in primary transverse balance.

(5) **Resultant Transverse Couples of Higher Order.**—By drawing I vector-polygons with their sides making angles three, five, seven . . . times those made by the cranks with the line of stroke, as described in sect. 5, Chap. VI., for *mr* vector-polygons, the resultant I of the orders 3, 5, 7 . . . can be obtained.

If an engine has an even number of cranks at equal angles, and if the cranks and connecting-rods are all similar, the resultant I's of all orders are zero, and the connecting-rods are in perfect transverse balance; as is evident also from the elementary consideration that at any instant the angular acceleration of pairs of connecting-rods are equal and opposite.

With engines having cranks in three axial planes at 120° (Fig. 11, Chap. III.; Figs. 18, 21, Chap. VI.), the I vector-polygons of the third, ninth, fifteenth . . . orders have their sides all lying in the same direction, and the corresponding transverse couples are unbalanced.

Example.—Find the maximum value of the transverse couple due to a locomotive connecting-rod; the data being as follows:—Stroke 26″, connecting-rod length between centres 78″, *i.e.* $q = \frac{1}{6}$, weight of connecting-rod 360 lbs., driving-wheels 6 feet diameter, speed 70 miles an hour.

$$\omega = \frac{70 \times 5280}{3600 \times 3} = 34{\cdot}2 \text{ radians per sec.}$$

I may be calculated if the dimensions of the connecting-rod are known; a rough approximation may be made as follows:—Taking the mass-centre at one-third the length measured from the big end, 240 lbs. mass may be taken at the big end, 120 lbs. at the small end. I is then, very roughly

$$240 \times \left(\frac{26}{12}\right)^2 + 120 \times \left(\frac{52}{12}\right)^2 = 3380 \text{ lb.-ft.}^2$$

The maximum value of the primary transverse couple is approximately

$$\mathrm{I}q\omega^2/g = \frac{34{\cdot}2^2 \times 3380 \times 1}{32{\cdot}2 \times 6} = 20{,}450 \text{ lb.-ft.}$$

Example.—In an engine for a motor-car, each connecting-rod weighs 8 lbs., the length between centres is $12\frac{1}{2}''$ stroke $= 5''$, *i.e.* $q = \frac{1}{5}$. Find maximum value of transverse couple due to one connecting-rod at a speed of 1500 revolutions per minute.

Take the value of I as 1 lb.-ft.^{2}, which will not be far from the truth, $\omega = 156$ radians per second. The maximum value of the primary transverse couple is approximately

$$\mathrm{I}q\omega^2/g = \frac{156^2 \times 1 \times 1}{32{\cdot}2 \times 5} = 151 \text{ lb.-ft.}$$

The maximum value of the third order couple is the above multiplied by $\frac{3}{8}q^2$, *i.e.*

$$\frac{151 \times 3 \times 1}{8 \times 25} = 2{\cdot}26 \text{ lb.-ft.}$$

In a two- or four-cylinder engine, with cranks at 180°, the primary and third orders transverse couples balance each other;

in a three-cylinder engine, with cranks at 120°, the primary couples balance, but those of the third order are added, the resultant maximum transverse couple being, for three equal connecting-rods as above, 6·78 lb.-ft. In the six-cylinder engine with cranks at 120°, the maximum transverse couple is twice the above, that is 13·56 lb.-ft.

CHAPTER VIII

ENGINES WITH CYLINDERS IN DIFFERENT LONGITUDINAL PLANES

(1) Most engines have the axes of the cylinders lying in the same plane passing through the axis of the crank-shaft; *e.g.* the usual types of stationary land engines, locomotives, and marine engines. A few types depart from this feature, the two-cylinder one-crank engine for motor-bicycles, the eight-cylinder petrol engine for motor-cars or launches, the "Brotherhood's" three-cylinder one-crank steam-engine, the four-cylinder one-crank hydraulic engine, and some paddle-wheel marine steam-engines.

(2) **Multicylinder One-crank Engines.**—In a one-crank multicylinder engine the axes of the cylinders are in a transverse plane, at right angles to the axis of the crank-shaft. Let the transverse plane be the plane of the paper (Fig. 1); O the centre of the crank-shaft; OC_1, OC_2, . . . the centre lines of the cylinders. Let OP be the crank in any position, then the crank-pin P has several connecting-rod ends jointed to it, the other ends reciprocating along OC_1, OC_2 . . . Of these only one is shown in the figure. The acceleration of each piston and the corresponding inertia force is along the axis of the cylinder or line of stroke. As discussed in Chap. VI., this acceleration or inertia force can be expressed as the sum of those due to simple harmonic motions of the first, second, fourth . . . orders. As discussed in Chap. IV., the simple harmonic motion of any order of a mass m, along the line of stroke is equivalent to two uniform circular motions in opposite directions of two masses each $\frac{1}{2}m$, the said two revolving masses coinciding with the

crank-pin on the line of stroke. The mass revolving in the same direction as the crank-pin may be called the equivalent *forward* mass, that revolving in the opposite direction the equivalent *reverse* mass. To get the resultant inertia force of the order k due to the n cylinders, there will be n forward masses and n reverse masses revolving at k times the speed of the crank. By the methods of Chap. III., the resultant of the n forward masses will be a single *resultant forward* mass, that of the n reverse masses a single *resultant reverse mass.*

(3) **Resultant Primary Forward Unbalanced Force.**—Let $m_1, m_2, \ldots$ be the masses at the respective cross-heads, then evidently the resultant primary forward mass is $\frac{1}{2}(m_1 + m_2 + \ldots)$ at the crank-pin, and revolves with it. This can always be balanced by a counterbalance mass opposite the crank.

(4) **Resultant Primary Reverse Unbalanced Force.**—Let the lines of stroke OC_1, OC_2, . . . make the angles $\beta_1, \beta_2, \ldots$ with any line of reference OX (Fig. 2). When the crank-pin is at P_0 on OX, the primary reverse mass $\frac{1}{2}m_1$ is at P_1'; the angles C_1OP_0 and C_1OP_1' being equal, that is, the angle XOP_1' is $2\beta_1$. Similarly, when the crank-pin is at P_0, the primary reverse mass m_2 is at P_2', the angle XOP_2' being $2\beta_2$; and so on for the other cylinders. Draw the mass vector-polygon (Fig. 3) with its sides of length $m_1, m_2 \ldots$ at angles $2\beta_1, 2\beta_2, \ldots$ with OX; OQ_0 being the resultant of the n mass vectors, $\frac{1}{2}OQ_0$ represents in magnitude the resultant primary reverse mass, and its position when the crank-pin is at P_0.

The primary reverse unbalanced force is therefore $OQ \times r\omega^2/2g$. This is a force constant in magnitude, and revolving in the opposite direction to the crank.

(5) **Resultant Forward and Reverse Unbalanced Forces of Higher Orders.**—Again, the simple harmonic motion of the kth order of a mass m is equivalent to two uniform circular motions (a forward and a reverse) of two masses each $\frac{1}{2}m$, revolving at k times the speed of the crank. When the crank is at OP_0, coincident with OX (Fig. 4), the forward and reverse masses of the kth order

have each rotated through an angle $k\beta$, from OC_1. That is, the forward mass is at P_1', the reverse mass at P_1''; the angle XOP_1' being $(k-1)\beta_1$, the angle XOP_1'' being

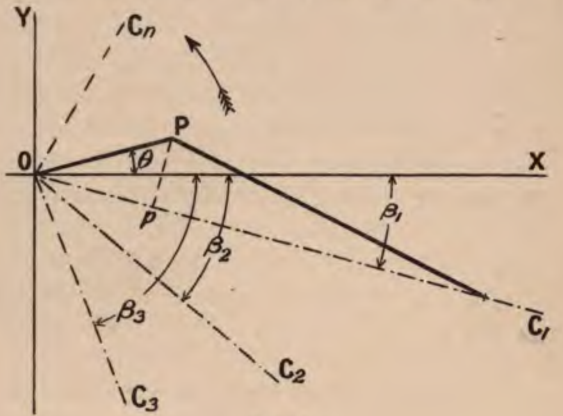

FIG. 1.

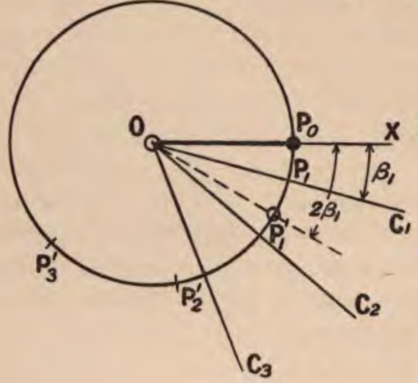

FIG. 2.

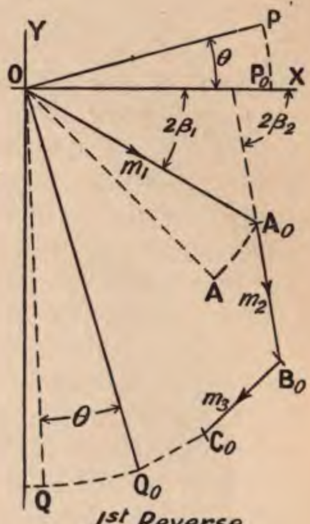

FIG. 3.

$(k+1)\beta_1$. Figs. 4–6 are drawn for the value $k = 2$, but the reasoning is the same for any value of k.

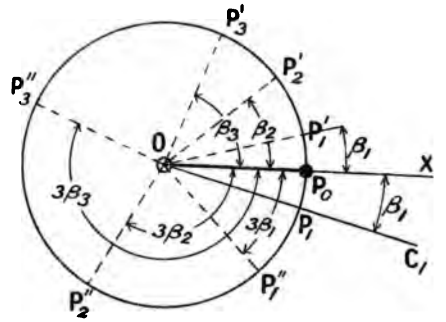

Fig. 4.

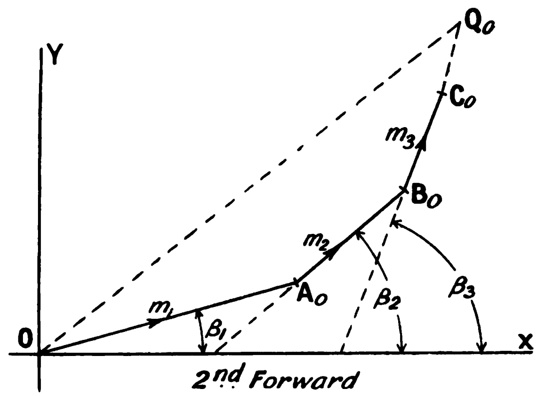

Fig. 5.

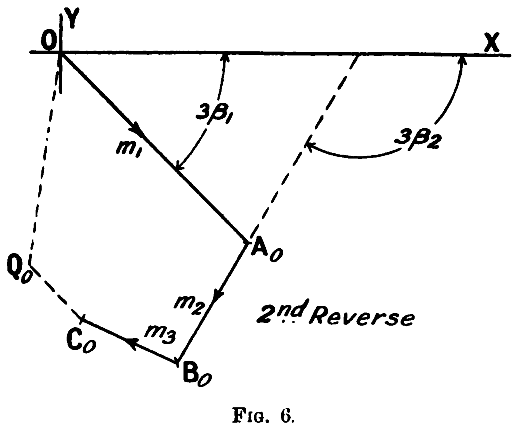

Fig. 6.

The resultant forward mass vector of the kth order, corresponding to crank at OP_0, is therefore obtained thus: Draw the mass-vector polygon (Fig. 5) with its sides of length m_1, m_2, . . . at angles $(k-1)\beta_1$, $(k-1)\beta_2$, . . . with OX in the forward direction of rotation, OQ_0 being the resultant of the mass vectors.

The resultant reverse mass vector OQ_0 of the kth order corresponding to crank at OP_0 is obtained by drawing the mass vector-polygon (Fig. 6) with its sides m_1, m_2, . . . at angles $(k+1)\beta_1$, $(k+1)\beta_2$, . . . with OX in the *reverse* direction.

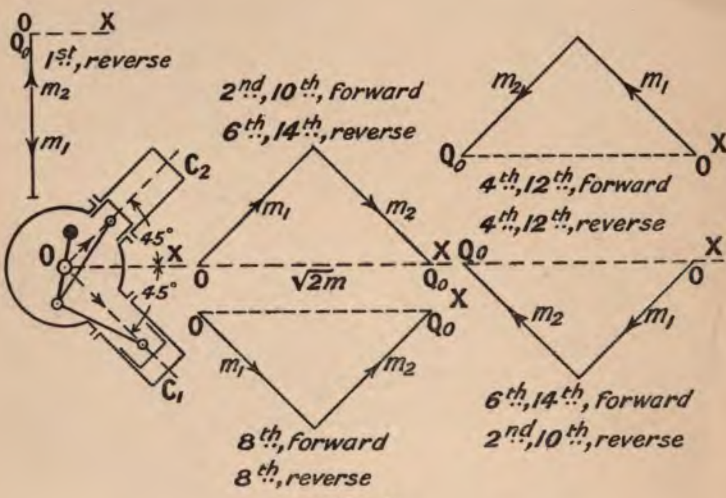

FIG. 7.

(6) **Example. Two Equal Cylinders at 90°.**—This type of engine is much used for motor bicycles and tricars. Fig. 7 represents it diagrammatically, and its mass vector-polygons. The primary reverse mass vector-polygon is closed, the primary balance is therefore perfect if the primary forward counterbalance mass is correct. The resultants OQ_0 of all other orders are each $\sqrt{2}m$, and the resultant mass vectors are each $m/\sqrt{2}$, m being the reciprocating mass for each cylinder. When the crank is in the direction OX bisecting the angle between the

cylinders, the resultant second forward mass vector is in the same direction, the resultant second reverse is in the opposite direction. The second forward and reverse rotating mass vectors have therefore a resultant always in a direction at right angles to OX (sect. 4, Chap. V.), its maximum value being $\sqrt{2}m$. Thus the secondary unbalanced force is a reciprocating one in the direction at right angles to OX, its greatest magnitude being approximately $\sqrt{2}mrq\omega^2/g$, or exactly $\sqrt{2}B_2A$, where $A = mr\omega^2/g$, and B_2 is the coefficient from Table V., Chap. VI.

Similarly, from the forward and reverse mass vector-polygons of the fourth order, it is easily seen that the unbalanced force of the fourth order reciprocates along OX, its maximum value being approximately

$$\sqrt{2}mrq^3\omega^2/4g,$$

or exactly, $\sqrt{2}B_4A$.

(7) **Even Number of Equal Cylinders at Equal Angles.**—The four-cylinder one-crank engine has been used

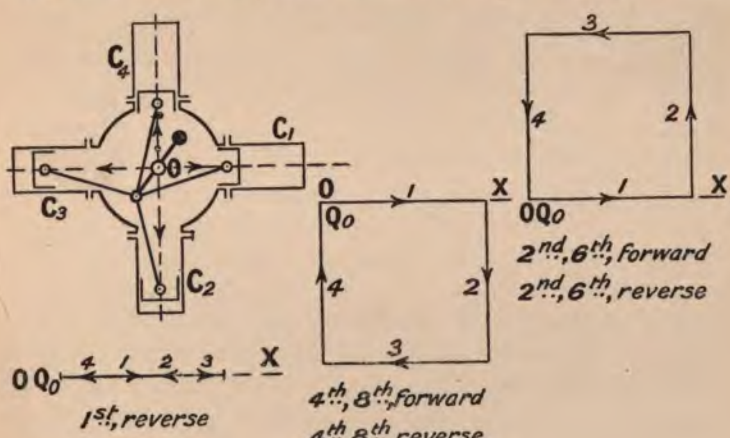

FIG. 8.

for hydraulic engines. It cannot be used for a gas or petrol engine working on the Otto four-stroke cycle, if the impulses are required to take place at equal intervals. Fig. 8 represents it diagrammatically, together with its mass vector-

I

polygons. It will be noticed that these are all closed, so that this engine has no unbalanced force of *any order*. The resultant transverse couple due to the angular swing of the connecting-rods being obtained by algebraic addition, the method is the same as discussed in Chap. VII. for the engine with cylinders in the same longitudinal plane, and with four cranks at 90°; in which case there is no resultant transverse couple of any order. Therefore, this engine is mathematically in perfect balance.

It is easily seen that the same is true for a one-crank

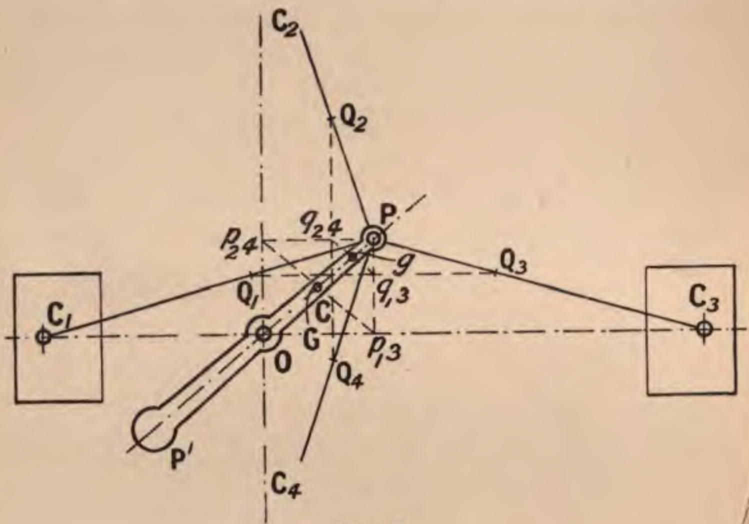

FIG. 9.

engine with any even number of cylinders at equal angles, provided the number of cylinders is greater than two. With two cylinders the primary reverse mass vector-polygon has the resultant $OQ_0 = 2m$, the mass vector-polygons of higher orders are all closed.

The perfect balance of the one-crank four-cylinder engine can easily be proved from elementary considerations. Fig. 9 shows it diagrammatically, with the crank-pin P in any position. From P drop the perpendiculars Pp_{13}, and Pp_{24} on to the two lines of stroke. Join $p_{13}\ p_{24}$, cutting the crank

OP at G. If the four pistons are exactly similar, and likewise the four connecting-rods, the joint mass-centre of the two pistons C_1 and C_3 is at p_{13}. Similarly, the joint mass-centre of the two pistons C_2 and C_4 is at p_{24}. It is easily seen that G is midway between p_{13} and p_{24}, therefore the joint mass-centre of the four pistons is at G. It is easily seen that G is midway between O and P. Therefore, the four pistons can be perfectly balanced by a counterbalance mass equal to that of two pistons placed at P′ at the same radius as the crank-pin P, in PO produced.

Let Q_1, Q_2, Q_3, Q_4 be the mass-centres of the four connecting-rods. Join Q_1 Q_3, cutting Pp_{13} at q_{13}; and join Q_2 Q_4, cutting Pp_{24} at q_{24}. Join q_{13} q_{24}, cutting OP at g. It is easily seen that g is a fixed point on the crank OP, and is the joint mass-centre of the four connecting-rods. The connecting-rod masses can therefore be perfectly balanced by a counterbalance mass M at P′, such that $M \times \overline{OP'}$ = mass of the four rods $\times \overline{Og}$. Thus, with a suitable counterbalance mass, the mass-centre of the system is always at O.

The angular swings of the two rods C_1P and C_3P are at any instant equal but opposite; their transverse couples, therefore, neutralize each other. Similarly for the two rods C_2P and C_4P.

(8) **Odd Number of Equal Cylinders at Equal Angles.** —The "Brotherhood" single-acting steam-engine, with three cylinders at 120°, and one crank, is a well-known example. The same arrangement is readily adaptable for a gas or petrol engine, working on the Otto cycle, as the impulses can take place at equal intervals. Probably five is the greatest number of connecting-rod ends that can be conveniently jointed to one crank-pin; it is possible that such an engine may be used, as its balance and other dynamical qualities are very good; and the type can be made much lighter than an engine with the same number of cylinders, and with a crank for each cylinder.

If n is the number of cylinders, at equal angles, the angle between two adjacent cylinders is $2\pi/n$, and if the masses m_1, m_2 . . . are all equal, the mass vector-polygons are

all equiangular and equilateral. The external angle of the primary reverse polygon is $4\pi/n$, of the second forward and reverse $2\pi/n$ and $6\pi/n$ respectively, of the fourth forward and reverse $6\pi/n$ and $10\pi/n$; and so on for higher orders. If either of the above angles is 2π, or a multiple of 2π, the n sides of the corresponding polygon are all in the same direction, the polygon is unclosed, and the resultant vector OQ_0 is nm. The other polygons are all closed. Thus

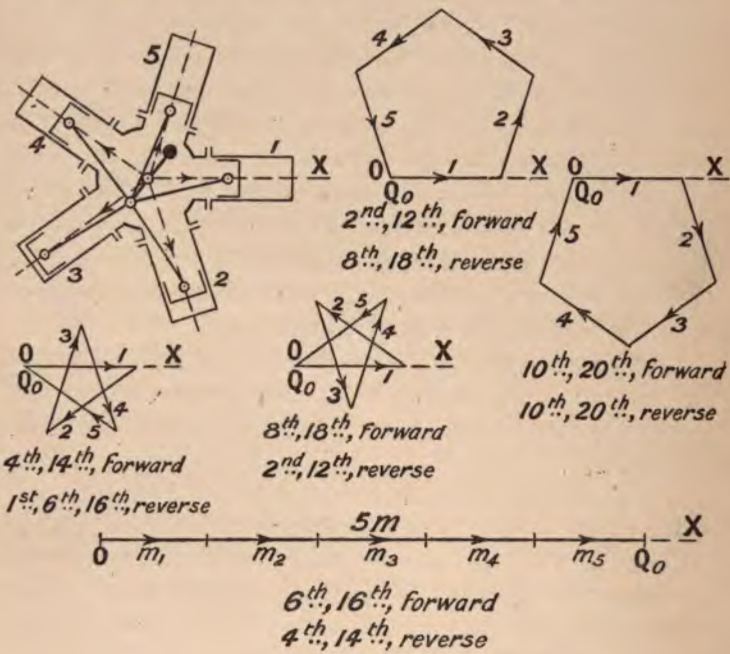

FIG. 10.

it is easily seen that all the vector-polygons are closed, except the reverse mass vector-polygons of the orders $(n-1)$, $(3n-1)$, $(5n-1)$, . . . and the forward mass vector-polygons of the orders $(n+1)$, $(3n+1)$, $(5n+1)$. . . .

The above discussion is illustrated by Fig. 10, which shows a five-cylinder one-crank engine, and the derived mass vector-polygons, which are all closed except the reverse of the orders 4, 14 . . . , and the forward of the orders

6, 16 . . . Hence, in this case, the largest unbalanced force, the reverse of the order 4, is approximately

$$\frac{5}{2}\frac{mr\omega^2}{g} \times \frac{1}{4}q^3 = \frac{5}{8}\ \frac{mrq^3\omega^2}{g},$$

or exactly, $\frac{5}{2}B_4A$.

(9) **Eight-cylinder Four-crank Engine.** — The eight-cylinder four-crank petrol engine, as used for motor-cars and launches, usually has the crank-shaft arranged as in Fig. 7, Chap. III., like that of the usual four-cylinder engine. The angle between a pair of cylinders is 90°, as in Fig. 7. With this arrangement the impulses take place at equal intervals, and a very even torque is obtained.

The primary balance for each crank with its pair of cylinders is perfect if a suitable counterbalance mass be used. Therefore, the eight cylinders and four cranks with four counterbalance masses, will also have the primary balance perfect. But the four equal counterbalance masses are balanced among themselves if the distances between the cranks are equal. The counterbalance masses may therefore be omitted without affecting the primary balance. Since the angle between the actual cranks is 180°, that between the imaginary secondary cranks is 360°. That is, the four imaginary secondary cranks are in line. Therefore, the resultant secondary unbalanced force is four times that for one crank, that is approximately $4\sqrt{2}mrq\omega^2/g$, or exactly, $4\sqrt{2}B_2A$. The same relation is true for the unbalanced forces of higher orders.

If the four cranks are set at 90° (Fig. 11), the angle between the imaginary secondary cranks is 180°. Thus the secondary balance is similar to the primary balance of a four-cylinder four-crank engine (Fig. 7, Chap. III.) running at twice the speed. Therefore, in this case, the secondary balance is perfect. To maintain the perfect primary balance the four counterbalance masses, or an equivalent system, represented by BB (Fig. 11), must be fastened to the crank-shaft.

To maintain the equal intervals between the impulses, the timing of the ignition in the various cylinders must be rearranged. The numbers shown in Fig. 11 indicate one

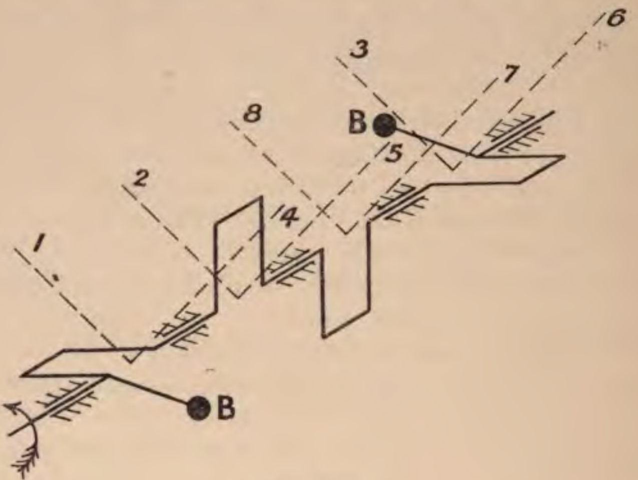

FIG. 11.

possible sequence of ignitions in the cylinders, giving impulses at equal intervals.

The largest unbalanced force will be that of the fourth order, of maximum value approximately $\sqrt{2}mrq^3\omega^2/g$, or exactly $4\sqrt{2}B_4A$.

(10) **Two-crank Multicylinder Petrol Engine.**—By taking two sets of the four-cylinder one-crank engine (Fig. 8) to form an eight-cylinder two-crank engine working on the Otto cycle, the impulses can be obtained at equal intervals. If the cylinders are placed side by side in pairs, the two cranks may be parallel, at 90° or 180°. The engine will of course be in perfect balance, if the primary counterbalance masses on the crank-shaft are properly arranged.

By taking two sets of a one-crank three- or five-cylinder engine (Fig. 10), a two-crank six- or ten-cylinder engine can be made. If the cranks are at 180°, the unbalanced forces all cancel out, leaving an unbalanced couple of the same order as the unbalanced force in the one-crank engine.

To keep the impulses at equal intervals, the two sets of cylinders should be "staggered," *i.e.* in a view looking parallel to the crank-shaft, the six or ten cylinders will be at equal angles of 60° or 36° respectively.

(11) **Marine Engine with Oblique Cylinders.**—Fig. 12 shows diagrammatically a type of engine sometimes used for paddle-wheel steamships. The crank-shaft being at a considerable distance above the water-line, to keep the mass-centre of the engine low, one cylinder, the low-pressure, is placed low down in the ship, the high-pressure cylinder above it. The axes of the cylinders are inclined, OC_1 and OC_2 (Fig. 12), the angle C_1OC_2 being as small as possible consistent with the dimensions of the cylinders. The angle

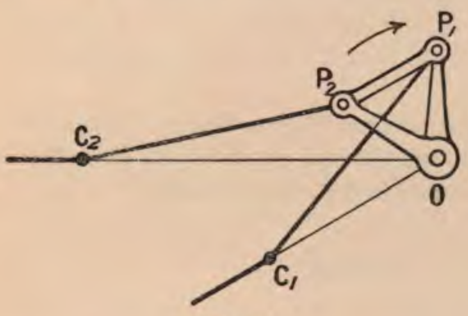

Fig. 12.

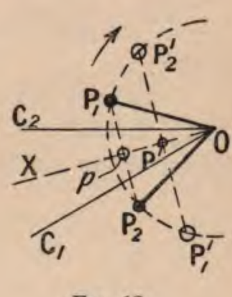

Fig. 13.

P_1OP_2 between the two cranks is the complement of the angle C_1OC_2; so that the torque on the shaft is similar to that in an engine with two coplanar cylinders, and cranks at 90°. The planes of motion of the connecting-rods are as close together as the conditions will admit. The two crank-pins are connected by a drag-link P_1P_2, and each crank has only a single cheek.

In Fig. 13, draw the centre lines OC_1 and OC_2 parallel to the lines in Fig. 12, and take an axis of reference OX bisecting the angle C_1OC_2. Draw the cranks OP_1 and OP_2 at equal angles with OX, as shown.

Considering the primary unbalanced forces, the mass m_1 reciprocating along OC_1 is equivalent to a forward and

reverse mass each $\frac{1}{2}m_1$, which coincide on the line OC_1. When the crank-pin P_1 is in the position shown, the reverse mass $\frac{1}{2}m_1$ is at P_1', the angle C_1OP_1' being equal to the angle C_1OP_1. Similarly, the corresponding position of the reverse mass $\frac{1}{2}m_2$ at P_2' is obtained. The resultant of the two reverse masses is a mass $\frac{1}{2}(m_1 + m_2)$ on the line $P_1'P_2'$. If $m_1 = m_2$, P' on OX is the position of the resultant reverse mass. The resultant forward mass is on the line P_1P_2, lying at p on OX if $m_1 = m_2$. This can be balanced by a counterbalance fixed opposite the line bisecting the angle P_1OP_2, in which case the primary unbalanced force will be of constant magnitude, rotating in the reverse direction. If no counterbalance mass is fitted, the resultant primary unbalanced force is in general elliptical, and is determined by the method of Fig. 5, Chapter V. If the angle C_1OC_2 is 45°, P_1O_2P is also 45°, and P' coincides with O; that is, the primary reverse force is zero.

There will be also a primary reverse couple, of magnitude $\frac{1}{2}P'P_1' \times ml\omega^2/g$. The unbalanced forces of the second and higher orders can be found in a similar manner.

Exercises.

1. In a V-type twin-cylinder motor-cycle engine; the angle between the cylinders is 60°; stroke, 80 millimetres; connecting-rod = 5 cranks; the reciprocating mass is $1\frac{1}{4}$ lb. for each cylinder, and the engine is half-balanced. Determine with an accuracy of 1 in 1000, the largest unbalanced force at a speed of 2500 revolutions per minute.

Ans. 1st reverse, 174·4 lbs.; 2nd forward, 61·0 lbs.; 2nd reverse and 4th forward, zero; 4th reverse, 0·59 lb.

2. If in the above engine the angle of the cylinders is 90°, determine the largest unbalanced force. *Ans.* 2nd, 99·60 lbs.; 4th, 0·99 lb.

3. In a three-cylinder one-crank motor-cycle engine the angles between the cylinders are 60° and 60°. The rest of the data being as in Example 1, determine the largest unbalanced force.

Ans. 2nd forward, 70·46 lbs.; 2nd reverse, 35·23 lbs.; largest 2nd order, 105·7 lbs.

CHAPTER IX

KINETIC ENERGY OF PISTONS AND CONNECTING-RODS

(1) **Kinetic Energy of Piston.**—Let OP and PC (Fig. 1) be the crank and connecting-rod in any position, and let OZ be drawn as in sect. 1, Chap. VI. The kinetic energy E_p

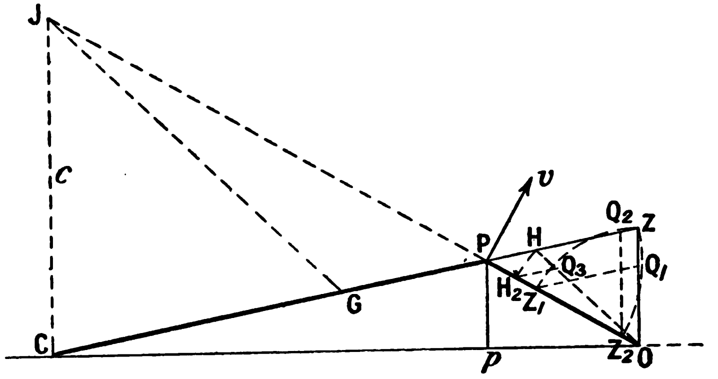

FIG. 1.

stored in the piston at the instant is $mV^2/2g$, m being the mass and V the speed of the piston.

Then since $\quad V = \dfrac{v}{OP} \cdot OZ, \; E_p = \dfrac{mv^2}{2g} \cdot \dfrac{OZ^2}{OP^2}$

In Fig. 1, mark off OZ_1 along OP equal to OZ, and draw Z_1Q_1 parallel to CP, cutting OZ at Q_1.

Then $\quad \dfrac{OZ}{OP} = \dfrac{OQ_1}{OZ_1}$, and $\dfrac{OZ^2}{OP^2} = \dfrac{OZ}{OP} \cdot \dfrac{OQ_1}{OZ_1} = \dfrac{OQ_1}{OP}$

Substituting above,

$$E_p = \frac{mv^2}{2g} \cdot \frac{OQ_1}{OP} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

If OQ be set off along OP equal to OQ_1, the locus of Q will be a polar curve showing the variation of the kinetic energy of the piston. Fig. 2 is the polar curve of kinetic energy of the piston for the case where the connecting-rod length is three and a half times the crank length. Fig. 3 is

FIG. 2.

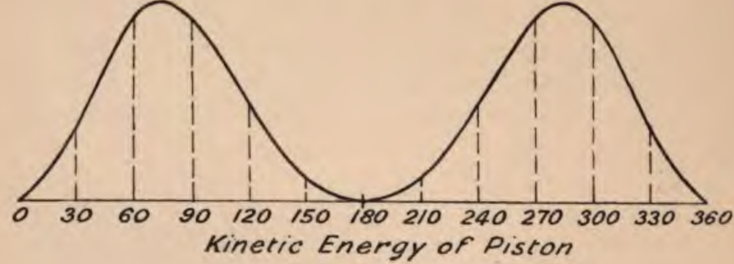

FIG. 3.

the corresponding rectangular curve, the abscissæ being the lengths of the path described by the crank-pin.

(2) **Kinetic Energy of Connecting-rod.**—The total kinetic energy of a rigid body may be expressed as the sum of its energies of translation and rotation. The kinetic energy of translation, or the *linear* energy, is the same as if the total mass of the body were concentrated at, and moved with, the same linear velocity as its mass-centre. Since the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on its sides, the linear energy of a body is the sum of its component linear energies in any two directions mutually at right angles. This principle is used in sect. 4 below, to find an algebraic expression for the linear

energy of the connecting-rod. The kinetic energy of rotation is the same as if the mass-centre of the body were fixed in space and the body rotated about an axis through the mass-centre parallel to the instantaneous axis with the same instantaneous angular speed.

Let G (Fig. 1) be the mass-centre of the connecting-rod, $I = mk^2$ its moment of inertia about an axis through G, parallel to the crank-shaft, k^2 being the square of its radius of gyration. Let V_g be the linear velocity of G. Then the kinetic energy of the connecting-rod is the sum of the linear energy of the mass m concentrated at G moving with speed V_g, plus the angular energy of the rod about its mass-centre. Thus, kinetic energy of connecting-rod

$$= \frac{mV_g^2}{2g} + \frac{I\omega_g^2}{2g} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where ω_g is the angular velocity of the connecting-rod.

Draw OH parallel to JG, cutting PZ at H (Fig. 1).

Since the points P and G on the connecting-rod are at the instant each rotating about the instantaneous centre J,

$$\frac{V_g}{v} = \frac{JG}{JP} = \frac{OH}{OP}$$

or

$$V_g = v \cdot \frac{OH}{OP}$$

$$V_g^2 = v^2 \cdot \frac{OH^2}{OP^2}$$

Set off OH_2 along OP equal to OH, and draw H_2Q_3 parallel to CP, cutting OH at Q_3.

Then $\quad \frac{OH}{OP} = \frac{OQ_3}{OH_2}$, and $\frac{OH^2}{OP^2} = \frac{OH}{OP} \cdot \frac{OQ_3}{OH_2} = \frac{OQ_3}{OP}$

Substituting above

$$V_g^2 = v^2 \cdot \frac{OQ_3}{OP}$$

Thus, linear energy of rod

$$E_l = \frac{mv^2}{2g} \cdot \frac{OQ_3}{OP}$$

Figs. 4 and 5 are respectively polar and rectangular curves showing variation of linear energy of the rod. Broadly speaking, the ordinates Fig. 5 may be said to be made up of a constant part plus a part which varies somewhat as in Fig. 3. The analogy is the same as considering the mass m at G to be replaced by two equivalent masses at P and C respectively, the mass at P having constant linear speed v, and therefore constant kinetic energy. If the mass

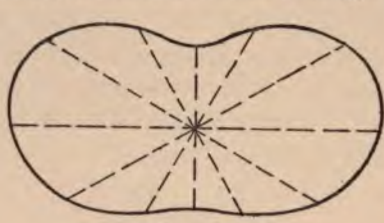

Fig. 4.

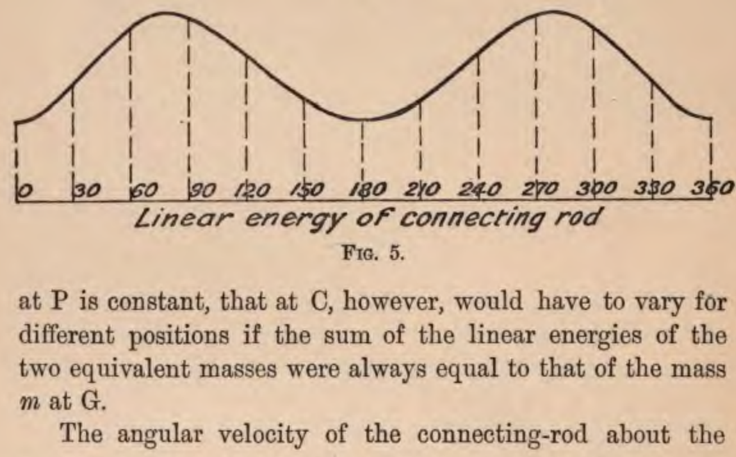

Fig. 5.

at P is constant, that at C, however, would have to vary for different positions if the sum of the linear energies of the two equivalent masses were always equal to that of the mass m at G.

The angular velocity of the connecting-rod about the instantaneous centre J is $\frac{v}{\mathrm{JP}}$.

The angular velocity of the crank is $\frac{v}{\mathrm{OP}}$. Therefore

$$\frac{\omega_g}{\omega} = \frac{v}{\mathrm{JP}} \cdot \frac{\mathrm{OP}}{v} \text{ or } \omega_g = \omega \frac{\mathrm{OP}}{\mathrm{JP}} = \omega \frac{\mathrm{PZ}}{\mathrm{PC}} = \frac{v}{r} \cdot \frac{\mathrm{PZ}}{l},$$

$$\text{since } \frac{\mathrm{PZ}}{\mathrm{PC}} = \frac{\mathrm{OP}}{\mathrm{JP}},$$

and $$\omega_g^2 = \frac{v^2}{l^2} \cdot \frac{PZ^2}{r^2}$$

Mark off PZ_2 along PO equal to PZ, and draw Z_2Q_2 parallel to OZ, cutting PZ at Q_2. Then

$$\frac{PZ}{PO} = \frac{PQ_2}{PZ_2}, \text{ and } \frac{PZ^2}{r^2} = \frac{PZ}{PO} \cdot \frac{PQ_2}{PZ_2} = \frac{PQ_2}{PO}$$

Therefore $$\omega_g^2 = \frac{v^2}{l^2} \cdot \frac{PQ_2}{PO}$$

Thus, angular energy of connecting-rod is

$$\frac{mk^2v^2}{2gl^2} \cdot \frac{PQ_2}{PO} = \frac{mv^2}{2g} \cdot \frac{k^2}{l^2} \cdot \frac{PQ_2}{PO}$$

Figs. 6 and 7 are respectively polar and rectangular

Fig. 6.

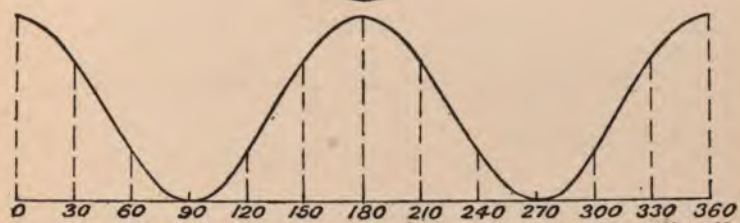

Fig. 7.

curves showing variation of angular energy of rod, which is, of course, a maximum when the crank is on the line of stroke.

(3) **Kinetic Energy of Piston. Analytical Method.**—The displacement of the piston when the crank has turned through the angle θ from the dead centre

$$x = r\cos\theta + l(A_0 + A_2\cos 2\theta + A_4\cos 4\theta + \ldots) \quad (1)$$

where the coefficients A_0, A_2, A_4, . . . have the values given in sect. 3, Chap. VI.

Also $$v = \frac{dx}{dt} = \frac{dx}{d\theta}\cdot\frac{d\theta}{dt} = \omega\frac{dx}{dt}$$

Therefore, by differentiating (1),

$$v = -\omega\{r\sin\theta + l(2A_2\sin 2\theta + 4A_4\sin 4\theta + \ldots)\} \quad (2)$$

or remembering that $l = r/q$,

$$v = -\omega r(\sin\theta + B_2\sin 2\theta + B_4\sin 4\theta + \ldots)$$

where
$$\begin{aligned} B_2 &= 2A_2 = \tfrac{1}{2}q + \tfrac{1}{8}q^3 + \tfrac{15}{256}q^5 + \ldots \\ B_4 &= 4A_4 = \quad -\tfrac{1}{16}q^3 - \tfrac{3}{64}q^5 - \ldots \\ B_6 &= 6A_6 = \qquad\qquad \tfrac{3}{256}q^5 + \ldots \end{aligned}$$

$$\therefore v^2 = \omega^2 r^2 \left\{ \begin{array}{l} \sin^2\theta + B_2{}^2\sin^2 2\theta + B_4{}^2\sin^2 4\theta + \ldots \\ + 2B_2\sin\theta\sin 2\theta + 2B_4\sin\theta\sin 4\theta + \ldots \\ + 2B_2B_4\sin 2\theta\sin 4\theta + \ldots \\ + \ldots \end{array} \right\} \quad (3)$$

By means of the well-known formula in Trigonometry,

$$2\sin A\sin B = \cos(A - B) - \cos(A + B),$$

(3) may be written

$$v^2 = \omega^2 r^2 \left\{ \begin{array}{l} \dfrac{1-\cos 2\theta}{2} + \dfrac{B_2{}^2}{2}(1-\cos 4\theta) + \dfrac{B_4{}^2}{2}(1-\cos 8\theta) + \ldots \\ + B_2(\cos\theta - \cos 3\theta) + B_4(\cos 3\theta - \cos 5\theta +) \\ + B_6(\cos 5\theta - \cos 7\theta) + \ldots \\ + B_2B_4(\cos 2\theta - 6\theta) + \ldots \\ + \ldots \end{array} \right\}$$

That is

$$v^2 = \omega^2 r^2 (C_0 + C_1 \cos\theta + C_2 \cos 2\theta + C_3 \cos 3\theta + \ldots) \quad (4)$$

where the coefficients have the following values :—

$$\left.\begin{array}{lllllll}
C_0 = \frac{1}{2}(1 + B_2^2 + \ldots) & = & \frac{1}{2} & +\frac{1}{8}q^2 & & +\frac{1}{16}q^4 & \quad + \\
C_1 = B_2 . & = & \frac{1}{2}q & & +\frac{1}{8}q^3 & & +\frac{15}{256}q^5 + \\
C_2 = \frac{1}{2} + B_2 B_4 + \ldots\ldots & = & -\frac{1}{2} & & & +\frac{1}{32}q^4 & \quad - \\
C_3 = -B_2 + B_4 . & = & -\frac{1}{2}q & & -\frac{3}{16}q^3 & & -\frac{27}{256}q^5 - \\
C_4 = \frac{1}{2}B_2^2 + \ \ldots\ldots & = & & -\frac{1}{8}q^2 & & -\frac{1}{16}q^4 & \quad - \\
C_5 = -B_4 + B_6 . & = & & & \frac{1}{16}q^3 & & +\frac{15}{256}q^5 + \\
C_6 = B_2 B_4 + \ldots\ldots\ldots & = & & & & \frac{1}{32}q^4 & \quad + \\
C_7 = -B_6 + B_8 . & = & & & & & -\frac{3}{256}q^5 -
\end{array}\right\} 5$$

When $\theta = 0$, the piston is at the beginning of its stroke, v is zero, and $\cos\theta$, $\cos 2\theta$, etc., are each equal to unity, and the sum of the coefficients C_0, C_1, C_2, must be zero, whatever be the value of q. The sum of the coefficients C_0, C_1 C_2, . . . can be expressed as a series with ascending powers of q, and as the sum is zero for all values of q, the coefficients of the various powers of q in the series must each be zero. This gives an easy check on the accuracy of the expressions for C_0, C_1, C_2. . . . The number of terms stated in each expression (5) is such as to enable a calculation to be made to an accuracy of 1 in 10,000. The following table gives values of C_0, C_1, C_2, . . . for various ratios of connecting-rod length to crank length, the range being wide enough to include most cases in practice, from an extremely short rod, to a very long rod, as found in some locomotives.

The kinetic energy of the piston of mass m is

$$E_p = \frac{mv^2}{2g} =$$

$$\frac{mr^2\omega^2}{2g}(C_0 + C_1 \cos\theta + C_2 \cos 2\theta + C_3 \cos 3\theta + \ldots) \quad (6)$$

TABLE VII.

COEFFICIENTS FOR KINETIC ENERGY OF PISTON.

$\frac{l}{r}=\frac{1}{q}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6	7
C_0	0·5106	0·5080	0·5063	0·5051	0·5035	0·5026
C_1	0·1459	0·1270	0·1125	0·1010	0·0839	0·0718
C_2	−0·5002	−0·5001	−0·5001	−0·5001	−0·5000	−0·5000
C_3	−0·1474	−0·1280	−0·1132	−0·1016	−0·0842	−0·0720
C_4	−0·0106	−0·0081	−0·0063	−0·0051	−0·0035	−0·0026
C_5	0·0016	0·0010	0·0007	0·0005	0·0003	0·0002
C_6	0·0002	0·0001	0·0001	0·0001	0·0000	0·0000

(4) **Linear Energy of Connecting-rod. Analytical Method.**—Let V_x and V_y be the components, respectively parallel and at right angles to the line of stroke, of the linear velocity V of the mass-centre of the connecting-rod. Then

$$V^2 = v_x^2 + v_y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and linear energy of rod, $E_l = \frac{mV^2}{2g} = \frac{m}{2g}(v_x^2 + v_y^2)$

Let $$\frac{CG}{CP} = a \text{ (Fig. 1)}; \quad \frac{GP}{CP} = b$$

Then $pP = r \sin \theta$, and the vertical component of P's velocity is $r \cos \theta \frac{d\theta}{dt} = \omega r \cos \theta$.

The vertical component of G's velocity is a times this value; therefore

$$v_y^2 = \omega^2 a^2 r^2 \cos^2 \theta = \frac{\omega^2 a^2 r^2}{2}(1 + \cos 2\theta) \quad . \quad (2)$$

The horizontal component of P's velocity is $-\omega r \sin \theta$. Let this be called v_p, and let v_c be C's velocity; then

$$v_x = av_p + bv_c \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$= -a\omega r \sin \theta - b\omega r(\sin \theta + B_2 \sin 2\theta + B_4 \sin 4\theta + \ldots)$$

Since $a + b = 1$,

$$v_x = -\omega r\{\sin\theta + b(B_2 \sin 2\theta + B_4 \sin 4\theta + \ldots)\} \quad (4)$$

$$\therefore v_x^2 = \omega^2 r^2 \begin{Bmatrix} \sin^2\theta + b^2B_2^2\sin^2 2\theta + b^2B_4^2\sin^2 4\theta + \ldots \\ +2bB_2\sin\theta\sin 2\theta + 2bB_4\sin\theta\sin 4\theta + \ldots \\ +2b^2B_2B_4\sin 2\theta\sin 4\theta + \ldots \\ + \ldots \end{Bmatrix} \quad (5)$$

$B_2, B_4, \ldots$ having the values stated in sect. 3.

But $V^2 = v_x^2 + v_y^2$, therefore adding (2) and (5) and transforming by the formula

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B),$$

$V^2 =$

$$\omega^2 r^2 \begin{Bmatrix} \frac{a^2}{2}(1+\cos 2\theta) + \frac{1}{2}(1-\cos 2\theta) + \frac{b^2B_2^2}{2}(1-\cos 4\theta) + \ldots \\ +bB_2(\cos\theta - \cos 3\theta) + bB_4(\cos 3\theta - \cos 5\theta) + \ldots \\ +b^2B_2B_4(\cos 2\theta - \cos 6\theta) + \ldots \\ + \ldots \end{Bmatrix} \quad (6)$$

That is,

$$V^2 = \omega^2 r^2(D_0 + D_1\cos\theta + D_2\cos 2\theta + D_3\cos 3\theta + \ldots) \quad (7)$$

where $D_0, D_1, D_2, \ldots$ have the following values :—

$$\left.\begin{aligned} D_0 &= \frac{a^2+1}{2} + b^2(\tfrac{1}{8}q^2 + \ldots) = \frac{a^2+1-b^2}{2} + b^2C_0 \\ D_1 &= b(\tfrac{1}{2}q \quad + \tfrac{1}{8}q^3 \quad + \ldots) = bC_1 \\ D_2 &= \frac{a^2-1}{2} - \frac{b^2q^4}{32} \quad - \ldots) = \qquad -ab + b^2C_2 \\ D_3 &= bC_3 \\ D_4 &= b^2C_4 \\ D_5 &= bC_5 \\ D_6 &= b^2C_6 \end{aligned}\right\} \quad . \quad (8)$$

where $C_1, C_3, \ldots$ have the values given in sect. 3.

The linear energy of the rod is therefore

$$E_l = \frac{mr^2\omega^2}{2g}(D_0 + D_1\cos\theta + D_2\cos 2\theta + D_3\cos 3\theta + \ldots) \quad (9)$$

The expression (9) is similar in form to that for the kinetic energy of the piston. The coefficients $D_0, D_1, \ldots$ can be quickly calculated from the coefficients C_0, C_1, C_2.

Taking $a = \frac{3}{5}$, $b = \frac{2}{5}$, Table VIII. gives the values of $D_0, D_1, D_2, \ldots$

TABLE VIII.

COEFFICIENTS FOR LINEAR ENERGY OF CONNECTING-ROD.

$\frac{l}{r} = \frac{1}{q}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6	7
D_0	0·6817	0·6812	0·6810	0·6808	0·6806	0·6804
D_1	0·0584	0·0508	0·0450	0·0404	0·0336	0·0287
D_2	−0·3200	−0·3200	−0·3200	−0·3200	−0·3200	−0·3200
D_3	−0·0591	−0·0512	−0·0453	−0·0406	−0·0337	−0·0288
D_4	−0·0017	−0·0013	−0·0010	−0·0008	−0·0006	−0·0004
D_5	0·0006	0·0004	0·0003	0·0002	0·0001	0·0001
D_6	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

In the case of the single-cylinder engine with the connecting-rod balanced (Fig. 1, Chap. V.), taking $q = \frac{1}{4}$, $a = \frac{6}{5}$, $b = -\frac{1}{5}$, the coefficients are

D_0	D_1	D_2	D_3	D_4
1·2203	− 0·0254	0·2200	0·0256	− 0·0003

(5) **Angular Energy of Connecting-rod. Analytical Method.**—Let α be the angular velocity of the connecting-rod when the crank is at the angle θ with the dead centre. Then E_a, the angular energy of the rod in gravitation units (foot-lbs.), is $\frac{I\alpha^2}{2g}$.

$$\alpha = \frac{d\phi}{dt} = \omega(A_1 \cos\theta + 3A_3 \cos 3\theta + 5A_5 \cos 5\theta + \ldots)$$

See Chap. VII., sect. 2.

$$\therefore \alpha^2 = \omega^2 \left\{ \begin{array}{l} A_1^2 \cos^2\theta + 9A_3^2 \cos^2 3\theta + 25A_5^2 \cos^2 5\theta + \ldots \\ + 6A_1A_3 \cos\theta \cos 3\theta + 10A_1A_5 \cos\theta \cos 5\theta + \ldots \\ + 30A_3A_5 \cos 3\theta \cos 5\theta + \ldots \\ + \ldots \end{array} \right\} \quad (1)$$

By means of the formula

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

(1) may be written

$$a^2 = \omega^2 \left\{ \begin{array}{l} \frac{A_1^2}{2}(1+\cos 2\theta)+\frac{9}{2}A_3^2(1+\cos 6\theta)+\frac{25}{2}A_5^2(1+\cos 10\theta)+\dots \\ +3A_1A_3(\cos 2\theta+\cos 4\theta)+5A_1A_5(\cos 4\theta+\cos 6\theta)+\dots \\ +15A_3A_5(\cos 2\theta + \cos 8\theta) + \dots \\ + \dots \end{array} \right\}$$

That is

$$a^2 = \omega^2(E_0 + E_2 \cos 2\theta + E_4 \cos 4\theta + E_6 \cos 6\theta + \dots) \quad (2)$$

and

$$E_a = \frac{I\omega^2}{2g}(E_0 + E_2 \cos 2\theta + E_4 \cos 4\theta + E_6 \cos 6\theta + \dots) \quad (3)$$

where the coefficients have the following values:—

$$\begin{aligned} A_1 &= q + \tfrac{1}{8}q^3 + \tfrac{3}{64}q^5 + \dots \\ A_3 &= \quad - \tfrac{1}{24}q^3 - \tfrac{3}{128}q^5 + \dots \\ A_5 &= \qquad\qquad \tfrac{3}{640}q^5 + \dots \\ &\dots \\ E_0 &= \tfrac{1}{2}A_1^2 + \tfrac{9}{2}A_3^2 + \dots \\ E_2 &= \tfrac{1}{2}A_1^2 + 3A_1A_3 + \dots \\ E_4 &= 3A_1A_3 + 5A_1A_5 + \dots \\ E_6 &= \tfrac{9}{2}A_3^2 + 5A_1A_5 + \dots \\ &\dots \\ i.e.\ E_0 &= \tfrac{1}{2}q^2 + \tfrac{1}{8}q^4 + \tfrac{1}{16}q^6 + \dots \\ E_2 &= \tfrac{1}{2}q^2 \qquad - \tfrac{1}{32}q^6 + \dots \\ E_4 &= \quad - \tfrac{1}{8}q^4 - \tfrac{1}{16}q^6 - \dots \\ E_6 &= \qquad\qquad \tfrac{1}{32}q^6 + \dots \end{aligned}$$

When $\theta = 0$, a the angular velocity of the rod is $\frac{v}{l} = \frac{\omega r}{l} = \omega q$, and $a^2 = \omega^2 q^2$; $\cos 2\theta, \cos 4\theta, \dots$ are then each equal to unity. A check on the accuracy of the expressions for E_0, $E_2 \dots$ is thus obtained. Again, when $\theta = 90°$, $\cos 2\theta = -1$, $\cos 4\theta = 1$, $\cos 6\theta = -1$. The angular velocity of the rod is then zero, therefore $E_0 - E_2 + E_4 - \dots = 0$, and another check is obtained.

Table IX. gives values for E_0, E_2, . . . for various values of q.

TABLE IX.

COEFFICIENTS FOR ANGULAR ENERGY OF CONNECTING-ROD.

$\frac{l}{r}=\frac{1}{q}$	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	6	7
E_0	0·0417	0·0317	0·0250	0·0202	0·0140	0·0102
E_2	0·0408	0·0312	0·0247	0·0200	0·0139	0·0102
E_4	−0·0009	−0·0009	−0·0003	−0·0002	−0·0001	0·0000
E_6	0·0000	0·0000	0·0000	0·0000	0·0000	0·0000

(6) **Average Energy.**—The sum of the first terms taken from each of the three expressions for the kinetic energy of the piston, linear and angular energies of the connecting-rod (these terms being independent of the angle θ, that is, independent of the crank-position) is

$$\frac{\omega^2}{2g}\{(m_p C_o + m_r D_o)r^2 + IE_o\} \quad . \quad . \quad . \quad (1)$$

m_p and m_r being the masses of the piston and connecting-rod respectively, I the moment of inertia of the connecting-rod. If to this is added the kinetic energy of the crank-shaft, with flywheel and all masses rotating with it, which is $\frac{I_s\omega^2}{2g}$, I_s being the moment of inertia of the crank-shaft and masses rotating with it, the average kinetic of the engine is

$$E = \frac{\omega^2}{2g}\{(m_p C_o + m_r D_o)r^2 + IE_o + I_s\} \quad . \quad (2)$$

The quantities inside the brackets are all constant, therefore (2) may be written

$$E = K\omega^2 \text{ foot-lbs.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

that is, the average energy of the engine is proportional to the square of the speed.

(7) **Excess of Energy, First, Second, Third . . . Orders.**—The other terms involving cos θ, cos 2θ, cos 3θ, etc.,

evidently show the fluctuation of energy of the mechanism as the crank-shaft revolves. They may be referred to as the excess energy, of the first, second, third . . . orders respectively, analogous to the expressions used for unbalanced forces. The value of a term, say $C_3 \cos 3\theta$ in the expression (6), sect. 3, for the kinetic energy of the piston for any value of θ, can evidently be regarded as the projection on the line of stroke of a vector of constant length, C_3, rotating at three times the speed of the crank, the rotating vector coinciding with the crank on its inner dead centre.

(8) **Multiple-cylinder Engines.**—When the engine has several cranks, the average kinetic energy of the mechanism is obtained by adding the values for the separate cranks and connecting-rods. The excess energy of orders 1, 2, 3 . . . are obtained by drawing vector-polygons, as discussed in Chaps. IV. and VI. If the cranks are at equal angles, and the pistons and connecting-rods are all equal, many of the terms will cancel out.

Fig. 8 represents a two-cylinder engine, cranks at 180°,

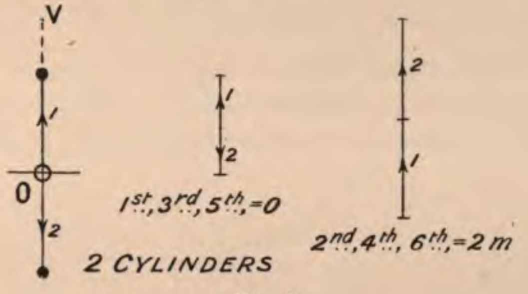

FIG. 8.

and the various vector-polygons. Those of orders 1, 3, 5, 7 . . . have zero resultant, those of orders 2, 4, 6, have resultant $2m$, m being the mass of one piston. Therefore, in this case

$$E_p = \frac{mr^2\omega^2}{g}(C_0 + C_2 \cos 2\theta + C_4 \cos 4\theta + \ldots) \quad (1)$$

Similar expressions hold for E_l, and E_a, the linear and angular energies of the connecting-rod.

The four-cylinder motor-car engine, with cranks all in

one plane, is also included in this case, the resultant vector being $4m$. Hence E_p is twice that stated above.

Fig. 9 is a similar diagram for a three-cylinder engine, cranks at 120°. The vector-polygons of orders 1, 2, 4,

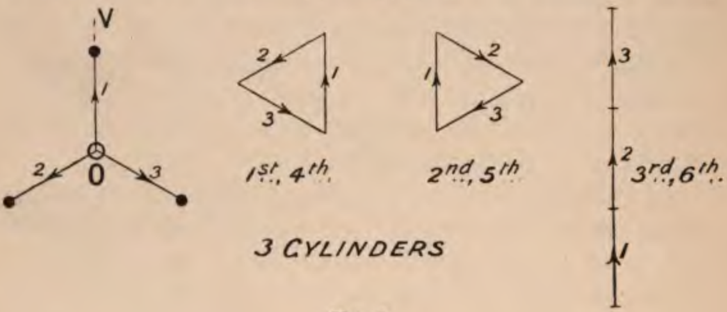

Fig. 9.

5 . . . have each resultant zero. Those of orders 3, 6 . . . have resultant $3m$. Hence, in this case

$$E_p = \frac{3mr^2\omega^2}{2g}(C_0 + C_3 \cos 3\theta + C_6 \cos 6\theta + \ldots) \quad (2)$$

$$E_a = \frac{3I\omega^2}{2g}(E_0 + E_6 \cos 6\theta + \ldots) . \quad . \quad . \quad . \quad . \quad (3)$$

For the six-cylinder motor-car engine with cranks at 120°, the values are twice those just written.

Fig. 10 is for a four-cylinder engine, cranks at 90°. The

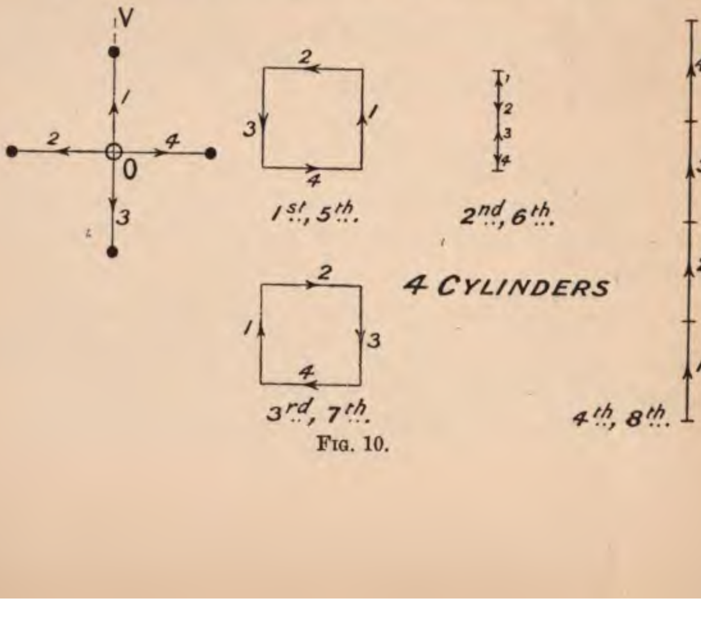

Fig. 10.

vector-polygons of the fourth, eighth . . . orders have resultant $4m$, the others zero.

Hence
$$E_p = \frac{2mr^2\omega^2}{g}(C_0 + C_4 \cos 4\theta + \ldots) \quad . \quad . \quad (4)$$
$$E_a = \frac{2I\omega^2}{g}(E_0 + E_4 \cos 4\theta + \ldots) \quad . \quad . \quad . \quad (5)$$

Fig. 11 is for a five-cylinder engine, cranks at equal angles. Here
$$E_p = \frac{5mr^2\omega^2}{2g}(C_0 + C_5 \cos 5\theta + \ldots) \quad . \quad . \quad (6)$$
$$E_a = \frac{5I\omega^2}{2g}(E_0 + E_{10} \cos 10\theta + \ldots) \quad . \quad . \quad (7)$$

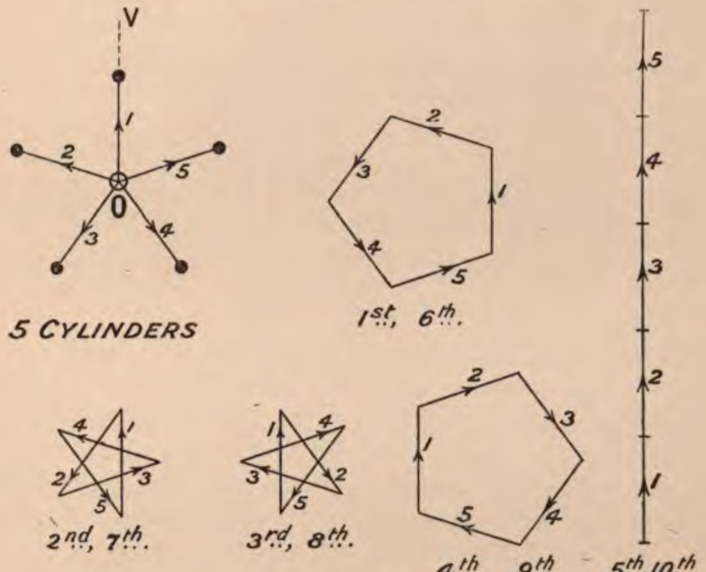

FIG. 11.

In Fig. 12, the curves 1, 2, 3, 4, show the fluctuation of kinetic energy of the piston or pistons in one-, two-, three-, and four-crank engines, cranks at equal angles, and piston masses equal, and connecting-rod equal to five cranks' length. The scale at the left hand corresponds to the factors outside the brackets in equations (6), sect. 3, and (1), (2), (4) above

(multiples of $mr^2\omega^2/2g$) being in each case equal to unity. The ordinates calculated from these equations are set downwards from the line AA. The ordinate to the straight line

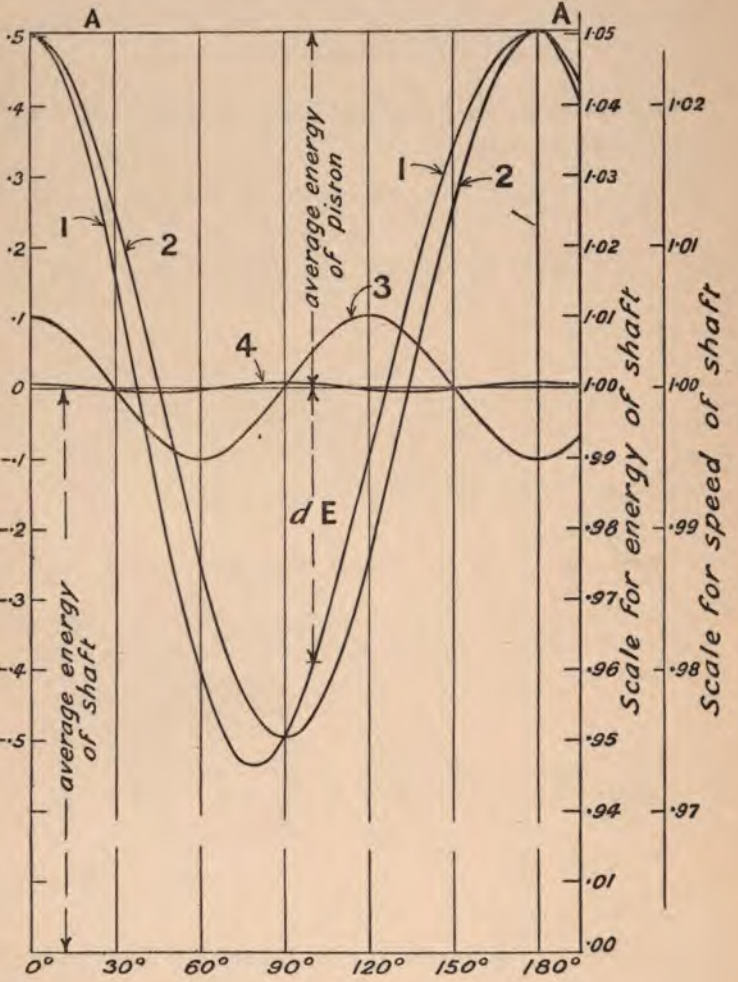

FIG. 12.

OO therefore measures the average kinetic energy of the pistons. From OO mark off downwards an ordinate equal to the average energy of the shaft, and draw the base line.

In general this will be much greater than the average energy of the pistons; the diagram is therefore broken off near the base. If at any point in the revolution the energy of the pistons is less than the average, that of the shaft is greater than the average. But the total kinetic energy is constant during the revolution. Therefore the ordinates measured upwards from the base line show the varying energy of the shaft.

The scale at the right-hand side for energy of shaft is drawn on the supposition that the I_s for the crank-shaft masses is ten times the sum mr^2 for the piston masses; and at the average speed of the shaft, ω_a, the energy of the shaft is marked unity.

Similar curves can be drawn taking account of the energy of the connecting-rods. Since in the single-cylinder engine the linear energy of the connecting-rod never has zero value (Fig. 5), in this case the line AA of total energy (Fig. 12) will lie above the curve 1.

(9) **Fluctuation of Speed.**—Let E be the energy of the crank-shaft with flywheel and other parts rotating with it, when the shaft is rotating at the average speed ω_a. Then

$$E = K\omega_a^2 \text{ ft.-lbs.} \quad . \quad . \quad . \quad . \quad (1)$$

Presuming we are dealing only with the inertia of the parts, that is, that the work done in the cylinders during any time interval is exactly equal to that required to overcome the resistance of the shaft, the total energy remains constant during the revolution, even although the speed of the shaft varies cyclically above and below the average. Let dE be the excess energy of the pistons and connecting-rods at any instant, above their average energy. This is obtained from the energy of the shaft, and the speed of the shaft falls to ω, below the average. Then if $(\omega_a - \omega)$ is small, neglecting differences of the second order

$$dE = K(\omega_a^2 - \omega^2) = K(\omega_a + \omega)(\omega_a - \omega) \quad . \quad (2)$$

Approximately, $(\omega_a + \omega) = 2\omega_a$, and (2) may be written

$$dE = 2K\omega_a(\omega_a - \omega) \quad . \quad . \quad . \quad . \quad (3)$$

Dividing (3) by (1), we get

$$\frac{(\omega - \omega_a)}{\omega_a} = -\frac{dE}{2E} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

that is, the fluctuation of speed is numerically half the fluctuation of energy. The scale for the speed of shaft in Fig. 12, in the vicinity of ω_a, is therefore twice that of the energy, and the same curves show the fluctuation of speed.

(10) **Examples.**—Taking $q = \frac{1}{5}$, the values of dE from sect. 8, and the values of the series of coefficients C, D, E, from Tables VII., VIII., and IX., we have from (4), for a two-cylinder engine, cranks at 180°

$$\frac{\omega - \omega_a}{\omega_a} = \frac{1}{2Kg}\{(0{\cdot}500m_p + 0{\cdot}320m_r)r^2 - 0{\cdot}020I\}\cos 2\theta \quad (9)$$

for a three-crank engine, cranks at 120°,

$$\frac{\omega - \omega_a}{\omega_a} = \frac{1}{2Kg}\{0{\cdot}152m_p + 0{\cdot}061m_r\}r^2 \cos 3\theta \quad . \quad . \quad (10)$$

for a four-crank engine, cranks at 90°

$$\frac{\omega - \omega_a}{\omega_a} = \frac{0{\cdot}010}{2Kg} m_p r^2 \cos 4\theta \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (11)$$

for a five-crank engine, cranks at 72°

$$\frac{\omega - \omega_a}{\omega_a} = -\frac{0{\cdot}0012}{2Kg} m_p r^2 \cos 5\theta \quad . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

with a degree of accuracy of one part in 100 for the two-crank engine, and greater as the number of cranks is increased.

The fluctuation of speed due to inertia of the mechanism alone does not vary with the average speed of the shaft. By putting $\cos\theta$, $\cos 2\theta$, . . . equal to unity in the above expressions, their greatest values are obtained.

For the four-, six-, and eight-cylinder motor-car engines respectively, the coefficients in the expressions for $(\omega - \omega_a)/\omega_a$ are twice those given in (9), (10) and (11); m_p, m_r, and I in each case referring to one piston or connecting-rod. For a

four-cylinder engine, taking $m_p = 4$ lbs., $m_r = 4$ lbs., $r = 2''$ $= \frac{1}{6}$ ft., $I = \frac{1}{2}$ lb.-ft.^{2}, then

$$2\{(0{\cdot}50m_p + 0{\cdot}32m_r)r^2 - 0{\cdot}02I\}$$

has the value 0·202 lb.-ft.^{2}. If the total average energy is equivalent to a mass of 64 lbs. rotating with the crank-shaft at a radius of 6 inches, $E = \dfrac{64 \times 1}{2 \times 4}\omega^2 = 8\,\omega^2$ foot-poundals, and $2Kg = 16$. The fluctuation of speed due to the inertia of the parts is then 0·012.

For a six-cylinder engine of the same power, and crank-shaft masses, taking $m_p = m_r = 3$ lbs., and substituting in (10), the fluctuation of speed due to inertia of the parts is 0·0013.

If gas-engines are to be applied on a large scale for marine purposes, heavy flywheels will be undesirable. Taking the extreme case that the kinetic energy of the masses revolving with the crank-shaft is negligibly small, the average kinetic energy is given by equation (1), sect. 6. The value of $2Kg$ for a single-cylinder engine is then (with the same assumptions as above)

$$(0{\cdot}505m_p + 0{\cdot}681m_r)r^2 + 0{\cdot}020I \quad . \quad . \quad (13)$$

From equations (10), (11), and (12) we get,

$$\text{for 3 cranks, } \frac{\omega_e}{\omega_a} = \frac{0{\cdot}051m_p + 0{\cdot}020m_r}{0{\cdot}505m_p + 0{\cdot}681m_r}$$

$$\text{for 4 cranks, } \frac{\omega_e}{\omega_a} = \frac{0{\cdot}0025m_p}{0{\cdot}505m_p + 0{\cdot}681m_r}$$

$$\text{for 5 cranks, } \frac{\omega_e}{\omega_a} = \frac{-0{\cdot}00025m_p}{0{\cdot}505m_p + 0{\cdot}681m_r}$$

where ω_e is the greatest variation above and below the average speed ω_a.

If $m_p = m_r$, the speed-fluctuations for the three-, four-, and five-crank engines are 0·060, 0·0021, and 0·00021 respectively. Thus, if the average speed is 100 revolutions per minute, the three-crank engine speed would vary from 94 to 106, the four-crank from 99·8 to 100·2, the five-crank from 99·98 to 100·02.

Exercises.

1. In a locomotive with two cranks at 90°, the stroke is 26 inches, connecting-rod length 91 inches centre to centre, reciprocating masses 400 lbs. for each cylinder, mass of each connecting-rod 450 lbs., mass-centre of connecting-rod 3 feet from big end, moment of inertia of connecting-rod 6000 lb.-ft.^{2} Calculate, with an accuracy of one part in 1000, the relative values of the average, least, and greatest kinetic energies of the mechanism relative to the engine frame. If the driving-wheels are 6 feet 6 inches diameter, and the speed of the locomotive 70 miles per hour, determine the absolute values of the average, least, and greatest kinetic energies.

Ans.—Relative, 1310·4, 1447·3, 1174·2 lb.-ft.^{2} Absolute, 40,610, 44,850, 36,390 ft.-lbs.

2. In examples 1, 2, and 3, Chap. VIII., each piston and gudgeon-pin weighs 14 ounces, each connecting-rod 18 ounces, mass-centre of rod at $\frac{3}{8}$th length from big end, moment of inertia $\frac{1}{12}$ lb.-ft.^{2} Determine the average, least, and greatest kinetic energies at a speed of 2500 revolutions per minute.

3. If in example 2 above, the moment of inertia of the flywheel in each case is $1\frac{1}{2}$ lb.-ft.^{2}, calculate the cyclical fluctuation of speed, when the spark is switched off, and the engine is running free.

CHAPTER X

TORQUE ON CRANK-SHAFT

(1) **Uniform Speed of Crank-shaft.**—In most engines, for whatever purpose used, a uniform angular speed of the crank-shaft is desired. The cyclic fluctuation of speed of the shaft depends on the nature of the resistance to be overcome, the cyclic fluctuation of the driving torque from the various cylinders, and the inertia of the parts, including the flywheel and all parts rigidly connected to the shaft, connecting-rods, pistons, valve gear. The resistance, or load, does not usually vary from instant to instant during a single revolution of the crank-shaft. The load may vary from time to time, in which case the power developed by the engine must also be varied, or excessive increase and decrease of speed of the crank-shaft will occur. To keep the fluctuation of the speed within narrow limits under wide variations of load is the function of the "governor." The discussion of this aspect of the problem does not lie within the scope of the present work. It will be assumed that the load or torque to be overcome by the crank-shaft is constant.

The torque on the shaft of a single-cylinder engine, whether water, steam, or gas, varies from instant to instant during one revolution. As far as the engine is concerned, the energy first exists in the required form of "mechanical work" in the cylinders. Up to that point, the energy has been latent or potential, either in the form of fluid (water, air, or steam) under pressure, or in the form of a mixture of combustible gases. At one instant the rate of generation of energy in the cylinders is greater than the rate of expenditure in overcoming

the internal frictional resistances and the external load, at another instant less. Applying the principle of the conservation of energy, the excess energy during any short period is expended in (*a*) increasing the potential energy of the system by doing work in elastic deformation of the mechanism, (*b*) increasing the kinetic energy of the system, thereby increasing the speed of the shaft. If the transmission mechanism is practically rigid, with nothing of the nature of a spring, the potential energy of the system remains unchanged, and the excess or deficiency of energy of the engine is wholly expended in increasing or decreasing the kinetic energy of the system.

(2) **Ratio of Tangential Effort at Crank-pin to Pressure on Piston.**—Let F be the pressure on the piston, f the corresponding *tangential* pressure on the crank-pin in the direction at right angles to the crank OP (Fig. 1, Chap. VI.). The tangential pressure f, multiplied by the crank-radius r, gives the *torque*, or twisting moment, on the shaft at the instant.

Neglecting frictional resistances, the work done on the piston during any small interval of time must be equal to the work transmitted to the crank-pin during the same interval. The forces at C and P at right angles to their actual motions at the instant, produce pressure on the slides and on the main-bearings respectively, but (neglecting friction) these forces do no work. Let dt be any very short interval of time, the distance traversed by the cross-head C in time dt is $\mathrm{V}dt$, the work done by the force F is $\mathrm{FV}dt$. Similarly, the work done at the crank-pin in time dt is $fvdt$. Therefore

$$\mathrm{FV} = fv. \quad \text{Or } f/\mathrm{F} = \mathrm{V}/v \quad . \quad . \quad . \quad . \quad (1)$$

Therefore, in Fig. 1, Chap. VI.,

$$\frac{\mathrm{F}}{f} = \frac{\mathrm{OZ}}{\mathrm{OP}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus, the same construction which served to determine ratio of the speeds at C and P serves to determine the ratio of the forces at P and C. Therefore, if the piston pressure F is constant throughout the stroke, as in a water-pressure

engine, the curve showing the varying torque on the shaft is the same as the curve showing the varying speed of the piston when the shaft rotates uniformly.

(3) **Curve of Torque, from Indicator Diagram.**—The pressure on the piston, however, is not usually the same throughout the stroke. In a steam-engine, the steam is cut off at a certain point, and the pressure falls during the remainder of the stroke. Similarly, in a gas-engine, the pressure begins to fall soon after the beginning of combustion. The varying pressure throughout the stroke is graphically recorded by the "indicator diagram." Fig. 1 is the indicator

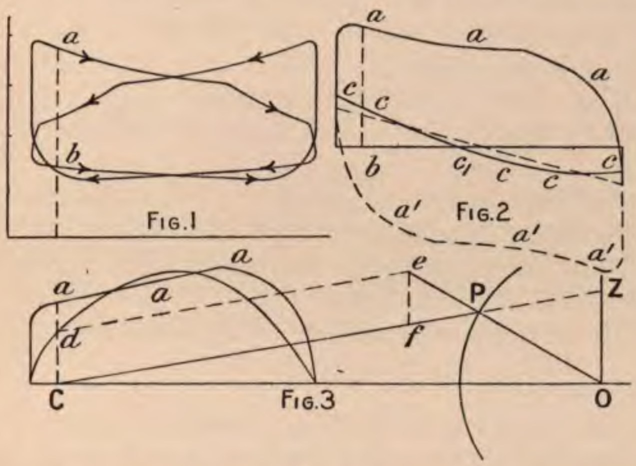

diagram from both ends of the high-pressure cylinder of a steam-engine, the arrows indicating the direction of motion of the piston. Drawing any ordinate, the intercept *ab*, between the steam line at *a* of the diagram for one end of the cylinder and the back-pressure line at *b* of the diagram for the other end, gives the effective pressure on the piston at that part of its stroke. These intercepts, *ba*, should be set off from a straight-line base (Fig. 2), giving the curve *aaa* of effective pressure on the piston. If the scale of the indicator diagram is one inch $= x$ lbs. per in.^{2}, and the area of the piston is A in.^{2}, one inch length of ordinate in Fig. 2 represents xA lbs effective pressure on the piston.

Fig. 2 is usually corrected for inertia of the reciprocating parts by plotting the curve of accelerating force, cc_1c. If the obliquity of the connecting-rod were neglected, this curve would be a straight line, shown dotted in Fig. 2, the ordinates at the two ends being $mr\omega^2/g$ lbs., m being the mass of piston, piston-rod, cross-head, and proportion of connecting-rod considered to be transferred to cross-head. The obliquity of the rod can be taken into account by drawing the Klein curve (Fig. 4, Chap. VI.), and reducing the ordinates, to correspond with the scale of the diagram. More conveniently, three points on the curve are quickly determined. The ordinate at the end furthest from the crank is $mr(1 + q)\omega^2/g$, at the end nearest the crank $mr(1 - q)\omega^2/g$, and the point c_1, of zero ordinate corresponds (with practical accuracy) to the piston position when the crank and connecting-rod are at right angles. A fair curve is then drawn through these three points.

At any ordinate bca, the ordinate bc to the inertia curve shows the force required to accelerate the reciprocating parts, for that piston position; the intercept ca is the force transmitted to the crank-pin.

The tangential effort on the crank-pin is ca multiplied by the ratio $\frac{\text{OZ}}{\text{OP}}$ (Fig. 3), and is most conveniently determined as follows: Draw the crank OP and connecting-rod PC in any position (Fig. 3). Set off the ordinate Ca equal to the intercept ca (Fig. 2) for the corresponding piston position. Produce OP, making Pe equal to Ca. Draw ed parallel to PC, cutting Ca at d. Cd is the required crank-pin effort.

For, draw OZ at right angles to CO, cutting CP, produced if necessary, at Z, and draw ef parallel to OZ. Then the triangles POZ and Pef are similar, and therefore

$$\frac{\text{C}d}{\text{C}a} = \frac{ef}{\text{P}e} = \frac{\text{OZ}}{\text{OP}}$$

The locus of d for different positions of the mechanism is the curve of crank-pin effort, with base corresponding to piston displacement.

If another curve (Fig. 4) be plotted, in which the abscissa is the linear distance moved by the crank-pin from the dead centre, and the ordinate is equal to Cd, the locus of d gives the rectangular curve of crank-pin effort to a time base, if the shaft rotates with uniform speed; and the area of the curve is equal to the work done in half a revolution.

The curve *ddd* for the next half-revolution is obtained by a similar process, taking the effective pressure during the backward stroke of the piston from the indicator diagrams (Fig. 1). The ordinates to the curve $a'a'$ may be set off downwards, as shown by the dotted line; the intercepts between this curve and the inertia curve, transferred to Fig. 3, being also set off downwards.

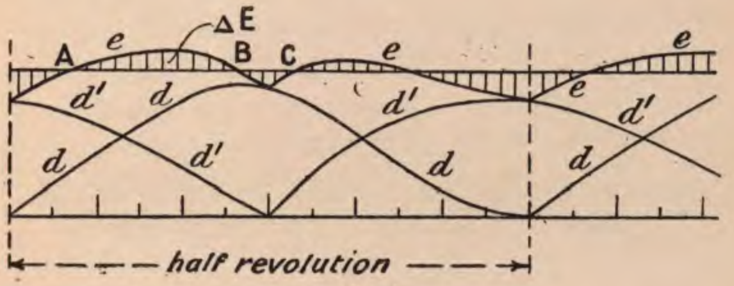

Fig. 4.

If the engine has two or more cylinders and cranks, the various crank-pin effort curves are drawn to the same base. In Fig. 4, the crank-pin effort curves for an engine with two cranks at right angles are shown. By suitably altering the scale of the ordinates, these curves may be regarded as curves of torque on the shaft. The ordinates to the curve *ee* (Fig. 4) are equal to the sum of the ordinates of the torque curves for separate cranks, and therefore give the resultant torque on the shaft. The horizontal line represents the average torque of resistance, the shaded areas the excess or deficiency of energy, or work done by the engine above or below that required to overcome the resistance. At the points A and B the effort and resistance are equal, but from A to B the effort is greater than the resistance, and the work represented

by the shaded area is expended in increasing the kinetic energy. The speed of the shaft therefore reaches a maximum value at B. From B to C the effort is less than the resistance, and the speed of the shaft diminishes, reaching a minimum value at C.

In any given engine, the excess energy, ΔE, the largest of the shaded areas (Fig. 4), can usually be expressed with sufficient practical accuracy as a fraction of the indicated work, R, done per revolution of the engine. In Fig. 4, this fraction is about 0·015.

Table X. gives a rough estimate of the values of this ratio for different types of engine, presuming that the indicator diagrams of all the cylinders are approximately similar, and that the impulses take place at equal angular intervals. This condition does not always hold with compound steam-engines, and therefore the values in the table must be used with caution.

TABLE X.

RATIOS OF EXCESS ENERGY ÷ AVERAGE ENERGY OF ONE REVOLUTION IN VARIOUS TYPES OF ENGINE.

Double-acting steam-engines.	Single-acting steam-engines and two-stroke cycle gas-engines.	Four-stroke cycle gas-engines.	$\frac{\Delta E}{R}$
Number of cylinders.			
		1	1·6 – 1·8
	1	2	0·80 – 1·10
		3	0·25 – 0·40
1	2	4	0·15 – 0·25
		5	0·07 – 0·12
	3	6	0·05 – 0·08
2	4	8	0·02 – 0·04
3	6	12	0·01 – 0·02
4	8	16	0·00 – 0·01

(4) **Fluctuation of Speed of Crank-shaft.**—Let E_a be the kinetic energy of the crank-shaft, flywheel, and any other parts keyed to it, when running at the average speed ω_a, ΔE the excess energy from the torque diagram, $\Delta\omega$ the difference between the maximum and minimum speeds of the crank-shaft. Then, as in sect. 9, Chap. IX., but remembering that here ΔE is the difference between greatest and least energies

$$\frac{\Delta\omega}{\omega_a}=\frac{1}{2}\frac{\Delta E}{E_a} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

E_a is equal to $\frac{I\omega_a^2}{2g}$, that is, E_a is proportional to the square of the speed of rotation. If the indicator diagrams are the same at different engine speeds, ΔE varies with the speed only in so far as the shape of the torque curve may be influenced by the different heights of the inertia curve (Fig. 2). Thus, we may say, with sufficient practical accuracy, that the fluctuation of speed, $\Delta\omega/\omega_a$, varies inversely as the square of the average speed of the engine.

(5) **Cyclic Variation of Speed of an Engine which has to run at Different Average Speeds.**—If the engine has to run at different speeds, and especially if it has two or more cylinders and cranks, the above method is tedious. This is the case for marine, locomotive, and motor-car engines. For different engine speeds the whole of the work must be repeated. Again, if the indicator diagrams are altered by throttling, varying the expansion, or otherwise, the work has to be repeated. In fact, to show the curve of torque for all conditions, two variables must be given independently a series of values. If, for example, curves of torque are required for four different speeds and for four different values of the average load, or torque (four different indicator diagrams), the drawings in Figs. 1–4 would have to be repeated sixteen times.

The following method is more expeditious. From the indicator diagrams, draw the curve of torque by the methods shown in Figs. 1–4, but without correction for the inertia of the reciprocating parts. Fig. 2 may therefore be omitted,

and the ordinates C*a* (Fig. 3) set off equal to the intercept *ba* (Fig. 1). The resultant torque curve is therefore independent of the speed of the engine, if the indicator diagrams do not alter with the speed. For different loads on the engine, if it be assumed that the indicator diagrams are similar, the curve of resultant torque will be the same for all loads, provided that the appropriate scale be used to read off the ordinates.

Fig. 5 shows an ideal indicator diagram from a gas-engine cylinder working on a four-stroke cycle, and a series of torque curves for engines with one, two, three, four, six, eight cylinders, but not corrected for inertia of reciprocating parts. In each case the curve to which the vertical ordinates are drawn is the torque curve; it will be noticed that with one, two, three, and four cylinders the torque is negative at one period. The ordinate to the horizontal line measures the average torque of resistance; the areas with diagonal shade lines give the excess or deficiency of energy ΔE. The other curve, in each case, is the graphic integral of the ΔE area; that is, the ordinate to this curve is proportional to the area dE up to that ordinate, and is therefore proportional to $(\omega - \omega_a)$, the excess or deficiency of speed above or below the average speed ω_a.

It will be noticed that as the number of cylinders is increased, the torque curve approximates more closely to a curve of sines; for eight cylinders the deviation from a sine curve not being very pronounced. If the torque curve for eight cylinders was an exact sine curve, that for sixteen cylinders, with impulses at regular intervals, would be a straight line, and ΔE would be zero.

Next, the curve of torque for the acceleration of the reciprocating parts is drawn. For a single-cylinder engine, this is done by applying the constructions of Figs. 3 and 4 to the inertia curve, Fig. 2. That is, by multiplying the ordinates of the Klein curve (Fig. 5, Chap. VI.), by the corresponding values of the ratio OZ/OP (Fig. 3), and choosing the scale to correspond with the mass and angular speed. Fig. 6 shows the Klein torque curve, as it may be called,

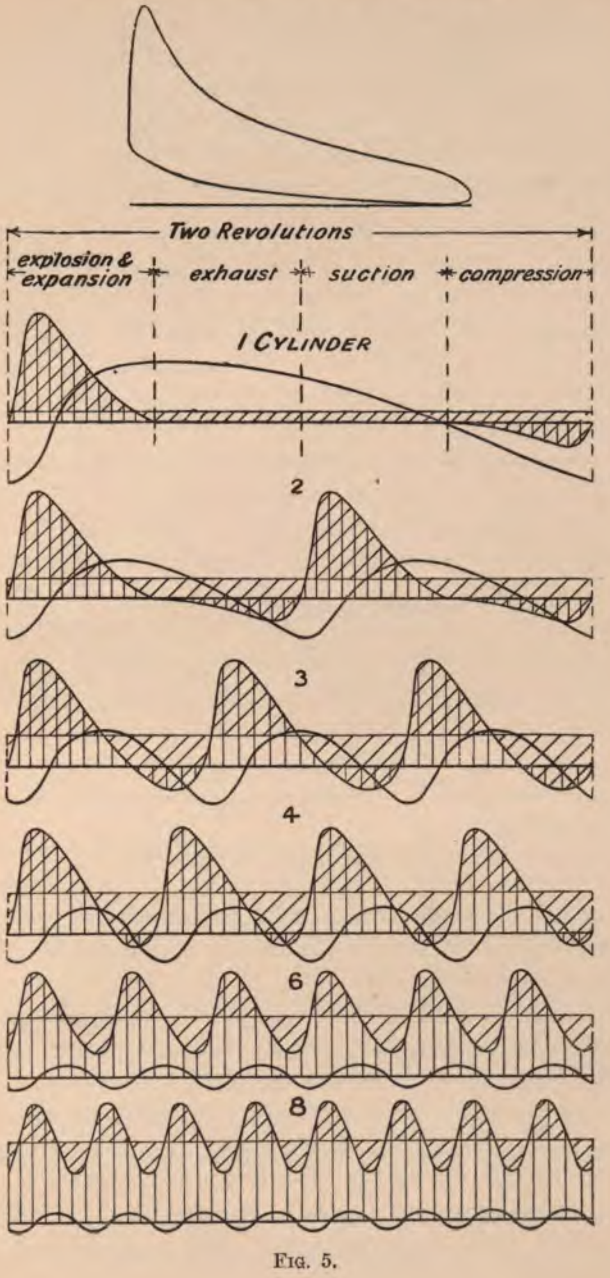
Two Revolutions
explosion & expansion
exhaust
suction
compression
1 CYLINDER
2
3
4
6
8

FIG. 5.

for a single cylinder, Figs. 7, 8, and 9, for engines with two, three, and four cranks respectively at equal angles and with equal reciprocating masses, the ordinates being drawn upwards when the crank-shaft is being accelerated.

The same method is applicable when crank angles and masses are unequal. The component torque curves are

Fig. 6.

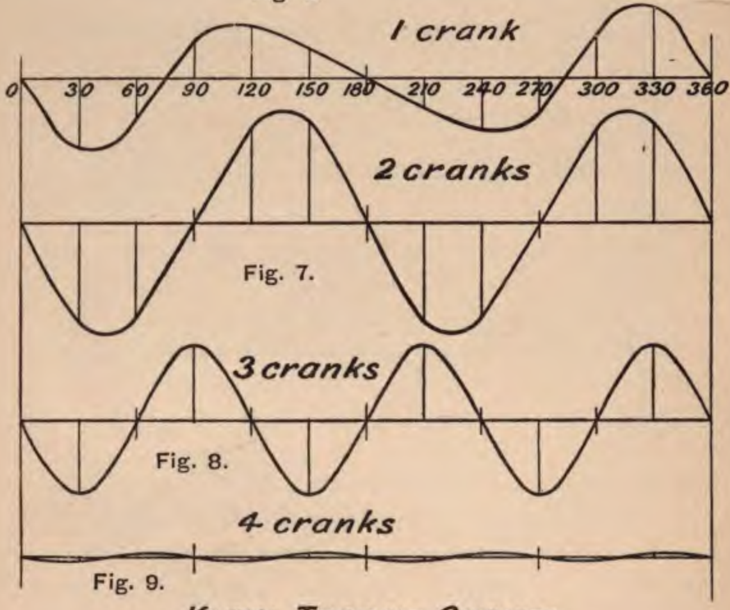

Fig. 7. Fig. 8. Fig. 9.

Klein Torque Curves

drawn each with its corresponding mass and proper displacement; the ordinates of the resultant Klein torque curve being got by algebraic addition.

The scale to be used in reading off the ordinates varies, of course, as the square of the crank-shaft speed. Thus, reading off, to the proper scale, the ordinates from the resultant torque curve drawn from the indicator diagrams typical of the engine, and from the Klein torque curve, a curve of torque for any load and any engine speed is quickly obtained.

(6) **Fluctuation of Speed of Multicylinder Gas-engine.**—If, as in a multicylinder petrol engine, the piston

masses and crank angles are equal, the fluctuation of speed can be quickly calculated, with sufficient practical accuracy for the comparison of different engines, without drawing curves of torque; all the necessary data, of course, being given.

For the four-, six-, and eight-cylinder engines, the curves showing dE, or $(\omega - \omega_a)$, approximate to sine curves (Fig. 5), the approximation being closer as the number of cylinders is greater. In each case the minimum ordinate is close to the beginning of the stroke, corresponding to the position when the actual torque is equal to the average torque. It is slightly later as the number of cylinders is increased. The maximum ordinate is evidently $\frac{1}{2}\Delta E$, ΔE being as defined in sect. 4. In one revolution of the engine the number of waves is 2, 3, and 4 for the four, six, and eight cylinders respectively. Taking the mean values of $\Delta E/R$ from Table X., the equations of these curves may be written—

For 4 cylinders, $$\frac{(\omega - \omega_a)}{\omega_a} = -\frac{0{\cdot}05R}{E_a}\cos 2\theta \quad . \quad . \quad . \quad (2)$$

For 6 cylinders, $$\frac{(\omega - \omega_a)}{\omega_a} = -\frac{0{\cdot}015R}{E_a}\cos 3\theta \quad . \quad . \quad . \quad (3)$$

For 8 cylinders, $$\frac{(\omega - \omega_a)}{\omega_a} = -\frac{0{\cdot}007R}{E_a}\cos 4\theta \quad . \quad . \quad . \quad (4)$$

the angle θ, in each case, being measured from the minimum ordinate.

Since $E_a = K\omega_a^2$, the fluctuation of speed due to uneven driving torque varies inversely as the square of the crank-shaft speed. It is greatest in the above cases when $\cos 2\theta$, $\cos 3\theta$, $\cos 4\theta$ are respectively equal to unity.

Example.—Comparing a four- and six-cylinder engine, each indicating 45 H.-P. at 1500 revolutions per minute, full load, and assuming the average kinetic energy of the engine to be equivalent to a mass of 64 lbs. at a radius of 6 inches ($\frac{1}{2}$ ft.), then

$$E_a = \frac{64 \times \omega^2}{2 \times 32 \times 4} = \frac{\omega^2}{4} \text{ ft.-lbs.}$$

R, the energy of a revolution at full load is—

$$\frac{45 \times 33{,}000}{1500} = 990 \text{ ft.-lbs.}$$

The fluctuations of speed due to uneven torque at different speeds and loads are shown, in each case, in the first line of Table XI.

TABLE XI.

FLUCTUATION OF SPEED OF ENGINE (45 H.-P. AT 1500 REVOLUTIONS PER MINUTE).

Revs. per min.	250	500	1000	1500
	Four-cylinder engine.			
Full load	0·300 0·288	0·075 0·063	0·019 0·007	0·008 – 0·004
$\frac{2}{3}$ load	0·200 0·188	0·050 0·038	0·013 0·001	0·006 – 0·006
$\frac{1}{3}$ load	0·100 0·088	0·025 0·013	0·006 – 0·006	0·003 – 0·009
	Six-cylinder engine.			
Full load	0·100 0·099	0·025 0·024	0·006 0·005	0·003 0·002
$\frac{2}{3}$ load	0·067 0·066	0·017 0·016	0·004 0·003	0·002 0·001
$\frac{1}{3}$ load	0·033 0·032	0·008 0·007	0·002 0·001	0·001 0·000

From the figures in the first line the fluctuation of speed due to the inertia of the pistons and connecting-rods have to be subtracted. These are 0·012 and 0·001 respectively for the four- and six-cylinder engines, at all speeds and loads. See Chap. IX., sect. 10. The results are shown in the second line in each case. Where a negative sign is shown, it indicates that at that speed the disturbance due to the

variation of kinetic energy of the pistons and connecting-rods is greater than that due to the unevenness of the driving torque.

(7) **Torque of Reaction.**—Newton's third law of motion is, "Action and reaction are equal and opposite." The pressure of the gases on the pistons produces a torque on the crank-shaft, and therefore at the same instant an equal opposite torque is produced on the frame of the engine. If the crank-shaft is fixed, as is the case with engines of the rotating-cylinder type, the engine frame revolves. If the engine frame is suspended on springs, it will, if the torque is uniform, be displaced from the normal position into a position of equilibrium. If the torque is variable, the engine frame will oscillate about the position of equilibrium. If a ship floats on an even keel when the engines are at rest, when driving full power ahead with a single propeller, the torque of reaction tends to capsize the ship, and the hull takes a list to such an angle that the righting moment is equal to the average torque on the crank-shaft. If the torque is variable, the hull oscillates slightly, the motion of oscillation about the mean position being determined by the equation $\Delta C = I\alpha$, where ΔC is the excess torque above the average, I the moment of inertia of the ship, about an axis through the mass-centre parallel to the crank-shaft, and α the angular acceleration about this axis.

On a motor-car, with propeller-shaft drive to a live axle, somewhat similar conditions hold, the chassis heeling sideways until the righting moment from the springs is equal to the torque on the shaft. But when the transmission is by side chains, the differential shaft and engine frame are fixed to the same rigid chassis; and the transverse couple communicated by the differential shaft to the chassis balances completely, or partially, the torque of reaction on the engine frame. When the differential gear is on the live axle, and the final drive to the differential is by a single chain, the conditions are similar to those in the latter case.

(8) **Function of the Flywheel.**—From the foregoing discussion, it is clear that the principal function of the flywheel

of an engine is to keep the cyclic variation of speed of the crank-shaft within prescribed limits. In conjunction with the governor, the flywheel also plays an important part in preventing great changes of the average speed when the load on the engine fluctuates; but that is a subject outside the scope of the present volume. The flywheel, clearly, cannot modify the variation of the torque of reaction on the engine frame.

In a locomotive, or motor-car, in which the whole vehicle is moving with a linear speed directly proportional to the angular speed of the crank-shaft, the mass of the whole vehicle acts as a flywheel, tending to keep the engine speed constant. No special flywheel is required on a locomotive. In a ship, this effect is modified by the varying "slip" of the propeller, the water not acting as a rigid "nut" for the "screw" to work in. In a motor-car, any relative slipping of the clutch surfaces or of the tyres on the ground, or any elastic yielding of the transmission gear, modifies the flywheel effect. The following short discussion is intended to show clearly the function of the flywheel of a motor-car engine.

Suppose, first, that the transmission system from engine to road surface is absolutely rigid, that there is no slipping of the clutch or of tyres, and no appreciable elastic yielding of shafts or of tyres in the circumferential direction of driving. These conditions are analogous to those obtaining on a rack-railway locomotive. Let the engine have a single cylinder, indicating 4 H.-P. at a crank-shaft speed of 400 revolutions per minute, when the car has a speed of 15 miles an hour (22 ft. per sec.). The average energy of the engine per revolution is

$$\frac{4 \times 33{,}000}{400} = 330 \text{ ft.-lbs.},$$

and taking the mean figure in Table VIII.,

$$\Delta E = 1{\cdot}7 \times 330 = 561 \text{ ft.-lbs.}$$

Let the total mass of the car and its load be 1600 lbs., then, neglecting at first the angular kinetic energy of road

wheels and engine, relative to the car frame, the average linear energy is

$$\frac{1600 \times 22^2}{2 \times 32} = 12{,}100 \text{ ft.-lbs.}$$

The fluctuation of speed is

$$\frac{\Delta v}{v} = \frac{1}{2}\frac{\Delta E}{E} = \frac{561}{2 \times 12{,}100} = 0{\cdot}023,$$

i.e. about $2\frac{1}{2}$ per cent. At a car speed of $7\frac{1}{2}$ miles an hour the fluctuation of speed would be about 10 per cent.

To appreciably reduce this fluctuation of speed by means of a flywheel, its angular energy should be greater than the linear energy of the whole car. Let the flywheel be equivalent to a mass of 120 lbs. at a radius of 9 inches. At the crank-shaft speed of 400 revolutions per minute, the linear speed of the flywheel rim is about 30 ft. per sec., the angular energy,

$$\frac{120 \times 30^2}{2 \times 32} = 1687 \text{ ft.-lbs.},$$

little more than one-eighth part of the linear energy of the car. Therefore, driving on the highest speed, the flywheel has little influence on the smoothness of the drive. But on the lowest speed, if the gear-ratio is 4 to 1, the angular energy of the flywheel would be sixteen times the amount stated above, more than twice the linear energy of the car. Thus the total kinetic energy of the system would be more than three times its value without the flywheel, and the fluctuation of speed would be reduced to one-third the former value. Elastic yielding and slipping of parts will modify the figures given above, but will not materially affect the general deduction.

Uneven torque on the crank-shaft causes a variation of the driving effort of the tyres at the road surface. The only external force which can cause the fluctuation of speed of the car is the tangential reaction of the road surface on the tyres. With no elastic yielding or slipping in the transmission

mechanism, the variation of driving-wheel effort is similar to that of crank-shaft torque. In the above example, on top gear the flywheel does not modify the driving-wheel effort to any great extent, but does so on lowest gear. Practically, therefore, the flywheel, however big within practical limits, has no great influence in equalizing the driving effort at the tyre surface. Hence, comparing the various curves of torque (Fig. 5, Chap. X.), the maximum effort on the tyre, with engines of one, two, four, six, eight cylinders, is respectively about 8, 4, 2, 1·4, 1·2 times the average.

On the other hand, a spring drive of proper design may render the driving effort on the tyres fairly uniform. The spring must be of ample volume, sufficient to absorb the excess energy of the engine (sect. 3, Chap. X.). The practice of dispensing with rigid torque rods, and transmitting the driving reaction of the live axle through the road-wheel springs, tends to equalize the driving effort on the tyres.

The principal function of the flywheel of a motor-car engine, therefore, is to give steady running of the engine when declutched from the transmission gear.

Exercises.

1. Describe the construction of a curve of velocity of the piston of a crank- and connecting-rod engine, and show that the same curve also represents the varying crank effort of the engine, supposing the pressure on the piston is constant. (Inst. C.E., Feb. 1905.)

2. A gas-engine running steadily has its load removed and the gas-supply cut off at the same instant. Assuming the speed to remain constant for a few seconds, what will be the stresses acting in that part of the crank-shaft between the flywheel and the crank due to the inertia of the piston? Sketch a curve of stress variation with crank-shaft angle. (Inst. C.E., Feb. 1906.)

3. The flywheel of an engine of 4 H.-P. running at 75 revolutions per minute is equivalent to a heavy rim 2 feet 9 inches mean diameter and weighing 500 lbs. Determine the maximum and minimum speeds of rotation when the fluctuation of energy is one-fourth the energy of a revolution. (Inst. C.E., Oct. 1900.) *Ans.* 84, 66 revs. per min.

4. In a gas-engine using the Otto cycle, the I.H.-P. is 8, and the speed is 264 revolutions per minute. Treating each fourth single stroke as effective and the resistance as uniform, find how many foot-pounds of energy must be stored in the flywheel in order that the speed shall not vary by more than one-fortieth of its mean value. (Inst. C.E., Oct. 1897.)

Ans. 30,000 ft.-lbs.

CHAPTER XI

PRIMARY AND SECONDARY BALANCE

(1) **Forces and Couples of Fourth Order small.**—In Chap. VI. it has been shown that in the single-cylinder engine the unbalanced forces of the first, second, fourth . . . orders rapidly decrease in magnitude as the order increases. Thus, if the connecting-rod length is five times that of the crank ($q = \frac{1}{5}$), the secondary unbalanced force is one-fifth part of the primary, the unbalanced force of the fourth order one-hundredth part of the primary. Therefore, if in a multiple-crank engine the primary and secondary forces and couples can be eliminated, the balance of the engine may be expected to be quite satisfactory.

It has been already shown (Chap. VI.) that a four-crank engine, with perfect primary and secondary balance, is theoretically possible, but the theoretical solution is impracticable, since two of the cranks are in the same plane. We proceed to consider the best possible balance with a practicable four-crank engine.

(2) **Four-crank Engine with Best Possible Balance.**—In the four-crank engine there are $(3n - 4) = 8$ independent variables. We have already seen (Chap. VI.) that for the primary and secondary forces and couples each to be zero, leads to an engine impracticable to realize. Let us impose the condition that the primary and secondary forces shall be balanced, likewise the primary longitudinal couple. That is, three vector-polygons shall be closed, which is equivalent to six equations of relation between the variables (sect. 6,

Chap. VI.), leaving two relations that may be specified at pleasure.

The balancing of the four-crank engine is discussed at length in "The Balancing of Engines," by Professor W. E. Dalby; here we will discuss that type in which the crank angles are symmetrical.

(3) **Symmetrical Cranks**.—Let us assume that (1) the angle between cranks 1 and 3 is equal to that between cranks 2 and 4. This condition is equivalent to saying that the crank-angle diagram (Fig. 1) is symmetrical; let VOV_1 be the trace of the plane of symmetry, OV being drawn coincident with the central plane of the cylinders, XOX_1 a line at right angles. Thus, with a four-crank symmetrical engine, one other condition may be assumed. It may be (2) either

(*a*) one angle θ_1, from the line OV is given,

(*b*) the ratio M_2/M_1 is given,

(*c*) the ratio l_{12}/l_{13} of two distances between two pairs of cranks is given.

Suppose the primary and secondary crank-angle diagrams to be drawn (Figs. 1 and 3), also the corresponding *mr* vector-polygons (Figs. 2 and 4). Then, from an inspection of these four figures, and a consideration of the relationship between them, it is evident that in order that the two *mr* vector-polygons be closed, a crank must lie in each of the four right angles at O. Therefore, if the angle VO1 (Fig. 1) is less than 45°, VO1 (Fig. 3) is less than 90°; thus, in Figs. 1 3, the cranks 1 and 4 lie above the line X_1OX; and, efore, cranks 2 and 3 lie below. In Fig. 3, the angle /3 measured counter-clockwise from OV is $>180°$, and 270°; therefore the angle VO3 (Fig. 1) is $> 90°$ and $< 135°$. urther, the condition that the cranks must be symmetrical necessitates that $m_4r_4 = m_1r_1$, and $m_2r_2 = m_3r_3$. For if not, the primary *mr* vector-polygon, if closed, must be unsymmetrical, as 1, 2′, 3′, 4′ (Fig. 2), with two sides, 3′ and 4′, longer than quasi-symmetrical sides 2′ and 1 respectively. If the sides are drawn in the same order for the primary and secondary *mr* vector-polygons, one of them must have a re-entrant angle. In Fig. 4, the sides 3′ and 4′ being longer

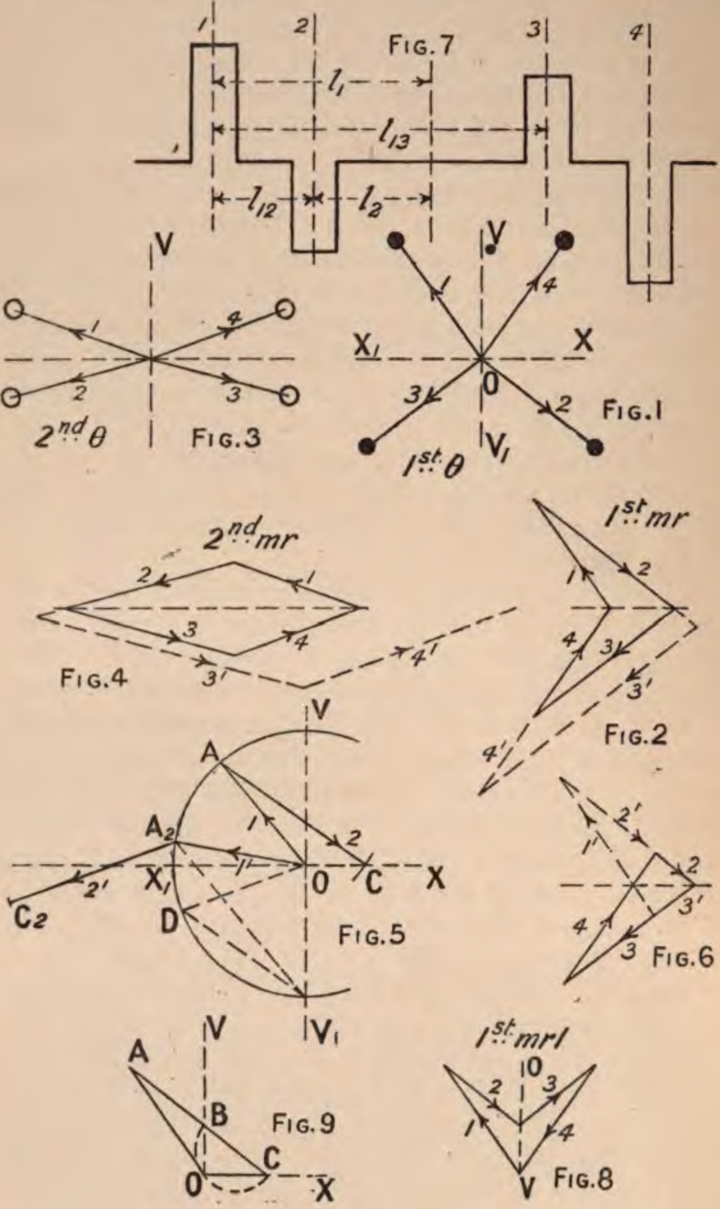
FIG. 7
l_1
l_{13}
l_{12}
l_2
1 2 3 4
FIG. 1
V
X₁ X
O
V₁
$1^{st}\theta$
FIG. 3
$2^{nd}\theta$
FIG. 4
$2^{nd}mr$
FIG. 2
$1^{st}mr$
FIG. 5
A A₂ C₂ X₁ D O C X V V₁
FIG. 6
FIG. 9
A B C O V X
FIG. 8
$1^{st}mrl$
O V

than 1 and 2, the vector-polygon would be unclosed. Therefore the sides 1 and 2 must be equal to the sides 4 and 3 respectively, in each *mr* vector-polygon, which are therefore symmetrical, as shown, about a line parallel to XOX_1.

(*a*) Assuming that the angle θ_1 is given, the solution may be completed by a trial-and-error method as follows:—

Draw a circle of radius m_1r_1 (Fig. 5), and draw diameters VV_1 and XX_1 at right angles. Draw OA, the *mr* vector for the crank 1, at the given angle θ_1 with OV. With centre A and any radius draw an arc cutting OX at C.

Draw the angle VOA_2 = twice the angle VOA. From A_2 draw A_2C_2 = AC, and making with OV twice the angle that AC makes. This is done most quickly by drawing V_1D parallel to CA, cutting the circle at D, and drawing A_2C_2 parallel to OD. If C_2 lies on X_1OX, AC is the correct value for the vector m_2r_2, if A_2 and C_2 lie on the same side of X_1OX, the assumed radius AC is too small, if on the opposite sides the assumed radius AC is too great. In the first event, OAC and OA_2C_2 are respectively halves of the two *mr* vector-polygons. If the angle θ_1 is less than 45°, AC is greater than OA; if θ_1 is greater than 45°, AC is less than OA.

The above construction is easily understood by the draughtsman who has no knowledge of trigonometry.

Draw now the primary *mrl* vector-polygon (Fig. 6) with respect to the transverse plane of the crank No. 1 (Fig. 7). The sides 2, 3, 4 are parallel to the corresponding sides of Fig. 2, the side 1 is zero. Since $m_2r_2 = m_3r_3$, the distances l_{12} and l_{13} of the planes of cranks Nos. 2 and 3 from the plane of crank 1, are proportional to the sides 2 and 3 of Fig. 6. The dotted triangle (Fig. 6) shows the *mrl* vector-polygon with respect to plane of crank 4, the side 4 being zero. Since $m_1r_1 = m_4r_4$, the side 1 of the dotted triangle is equal to the side 4 of the full-line triangle (Fig. 6); and from the symmetry of the *mr* polygon, it is evident that the side 2 of one triangle is equal to the side 3 of the other. Therefore the distance between cranks 1 and 2 is equal to that between 3 and 4. Thus, the symmetrical crank-angle diagram (Fig. 1) necessitates a symmetrical longitudinal diagram (Fig. 7); and

if l_{12} be given equal to l_{34}, this is not an independent relation between the variables. The converse, however, is not necessarily true.

(*b*) Assuming now that the ratio M_2/M_1 is given, the procedure is somewhat similar. If M_1 is less than M_2, draw the trial vector OA at an angle less than 45° with OV (Fig. 5). The length of the radius AC is known, and the construction of Fig. 5 can be completed.

(*c*) Let the ratio l_2/l_1 of the distances of cranks 2 and 1 from the plane of symmetry (Fig. 7) be given. Suppose the *mrl* vector-polygon drawn (Fig. 8) with respect to this plane of symmetry. Since $l_1 = -l_4$, and $l_2 = -l_3$, the polygon will be symmetrical about OV. Combining the *mr* and *mrl* vectors for cranks 1 and 2 in one diagram (Fig. 9), OX and OV are at right angles, as in Fig. 1, OA and ABC are parallel to cranks 1 and 2 respectively. B and C lie on OV and OX respectively. OA represents the vector m_1r_1, and also the vector $m_1r_1l_1$, which is equivalent to assuming l_1 equal to unity. AC is the vector m_2r_2, AB the vector $m_2r_2l_2$; therefore $l_2 = AB/AC$, or $l_2/l_1 = AB/AC$.

Taking the points A, B, and C along the straight line, such that $AB/AC = l_2/l_1$, on BC as diameter construct a semicircle, which will be the locus of O. Take a trial point, O, and complete the construction of Fig. 5. When C_2 falls on XOX_1 (Fig. 5), OA/OC (Fig. 9) is equal to m_1r_1/m_2r_2.

(4) **Symmetrical Cranks. Analytical Method.**—Let m_1, m_2, m_3, m_4 be the lengths of the sides of a closed vector-polygon, making angles θ_1, θ_2, θ_3, θ_4 respectively with any line of reference, OV. Let $x = \cos\theta$, $y = \sin\theta$, then

$$x^2 + y^2 = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also
$$\cos 2\theta = \cos^2\theta - \sin^2\theta = (2x^2 - 1) \quad . \quad . \quad (2)$$
$$\sin 2\theta = 2\sin\theta\cos\theta = 2xy \quad . \quad . \quad . \quad . \quad (3)$$

Since the sum of the projections on the reference line is zero

$$m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 0 \quad . \quad . \quad (5)$$

Similarly, for the projections at right angles

$$m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 = 0 \quad . \quad . \quad (6)$$

If the above is the primary mr vector-polygon, and the secondary vector-polygon is also closed, we have two similar equations in $2\theta_1$, $2\theta_2$, $2\theta_3$, $2\theta_4$.

$$\begin{aligned} m_1(2x_1^2 - 1) + m_2(2x_2^2 - 1) + m_3(2x_3^2 - 1) \\ + m_4(2x_4^2 - 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7) \\ m_1x_1y_1 + m_2x_2y_2 + m_3x_3y_3 + m_4x_4y_4 = 0 \quad . \quad (8) \end{aligned}$$

If the primary mrl vector-polygon is closed, we have two equations in the ml's similar to (5) and (6) in the m's

$$m_1l_1x_1 + m_2l_2x_2 + m_3l_3x_3 + m_4l_4x_4 = 0 \quad . \quad (9)$$
$$m_1l_1y_1 + m_2l_2y_2 + m_3l_3y_3 + m_4l_4y_4 = 0 \quad . \quad (10)$$

Similarly, if the secondary mrl vector-polygon is closed

$$\begin{aligned} m_1l_1(2x_1^2 - 1) + m_2l_2(2x_2^2 - 1) + m_3l_3(2x_1^2 - 1) \\ + m_4l_4(2x_4^2 - 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11) \\ m_1l_1x_1y_1 + m_2l_2x_2y_2 + m_3l_3x_3y_3 + m_4l_4x_4y_4 = 0 \quad . \quad . \quad . \quad (12) \end{aligned}$$

For a four-crank symmetrical engine we have seen that

$$m_1 = m_4, \; m_2 = m_3$$
$$\theta_1 = -\theta_4, \text{ therefore } x_1 = x_4, \; y_1 = -y_4$$
$$\theta_2 = -\theta_3, \text{ therefore } x_2 = x_3, \; y_2 = -y_3$$

Also, if the l's are measured from the transverse plane of symmetry of the crank distances

$$l_1 = -l_4, \quad l_2 = -l_3$$

Substituting these values in (5), (7), (10), they become

$$m_1x_1 + m_2x_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$
$$m_1(2x_1^2 - 1) + m_2(2x_2^2 - 1) = 0 \quad . \quad . \quad (14)$$
$$m_1l_1y_1 + m_2l_2y_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (15)$$

while the terms on the left-hand sides of the equations (6), (8), (9) cancel each other.

Equations (13), (14) and (15) are homogeneous in m_1 and m_2, they therefore involve the ratio m_2/m_1. Equation (15) is homogeneous in l_1 and l_2, and therefore involves the ratio l_2/l_1. These three equations therefore involve four quantities,

m_2/m_1, l_2/l_1, x_1, and x_2. If any one of these four quantities be given, the other three can be determined.

From (13) $$\frac{m_2}{m_1} = -\frac{x_1}{x_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

Substitute in (14) $x_2(2x_1^2 - 1) - x_1(2x_2^2 - 1) = 0$

or $$(x_1 - x_2) + 2x_1x_2(x_1 - x_2) = 0$$

From which, since $(x_1 - x_2)$ is not necessarily zero

$$x_1x_2 = -\tfrac{1}{2}, \text{ or, } x_2 = -1/2x_1 \quad . \quad . \quad . \quad (17)$$

That is $$\cos\theta_1 \cos\theta_2 = -\tfrac{1}{2}$$

Combining (16) and (17) $m_2/m_1 = 2x_1^2 \quad . \quad . \quad . \quad . \quad (18)$

From (15) $$\frac{l_2}{l_1} = -\frac{m_1y_1}{m_2y_2}$$

$$\therefore \left(\frac{l_2}{l_1}\right)^2 = \frac{(1 - x_1^2)}{4x_1^4(1 - x_2^2)} = \frac{(1 - x_1^2)}{x_1^2(4x_1^2 - 1)} \quad . \quad (19)$$

Write L for l_2/l_1, then (19) may be written

$$L^2 = \frac{(1 - x_1^2)}{x_1^2(4x_1^2 - 1)}, \quad i.e. \ (4x_1^4 - x_1^2)L^2 = (1 - x_1^2)$$

$$4L^2x_1^4 - x_1^2(L^2 - 1) - 1 = 0$$

Therefore

$$x_1^2 = \frac{(L^2 - 1) \pm \sqrt{(L^2 - 1)^2 + 16L^2}}{8L^2}$$

$$= \frac{L^2 - 1 \pm \sqrt{L^4 + 14L^2 + 1}}{8L^2} \quad . \quad . \quad (20)$$

(*a*) If θ_1 is given, x_2 can be calculated from (17), and θ_2 is known. m_2/m_1 can be calculated from (18), l_2/l_1 from (19).

(*b*) If m_2/m_1 is given, x_1 is calculated from (18), then x_2 from (17), and l_2/l_1 from (19).

(*c*) If the ratio l_{12}/l_{13} is given, L is known, and x_1 can be calculated from (20); then x_2 from (17), and m_2/m_1 from (18).

(5) **Magnitude of Secondary Couple.**—Fig. 10 shows the second *mrl* vector-polygon, OQ_0 being the maximum value of the resultant *mrl*. The components of OQ_0 parallel to OV and OX respectively, are given by the items on the left-hand sides of equations (11) and (12). Substituting the

equalities defining the symmetrical four-crank engine, the left-hand side of (11) becomes zero, that of (12) becomes

$$\begin{aligned} OQ_0 &= 2(m_1l_1x_1y_1 + m_2l_2x_2y_2) \\ &= m_1l_1y_1\left(2x_1 + \frac{1}{x_1}\right) \\ &= m_1l_1(\sin 2\theta_1 + \tan \theta_1) \end{aligned}$$

Taking $\theta = 30°$, $OQ_0 = 1{\cdot}443m_1l_1$, and the maximum value of the couple is

$$1{\cdot}443m_1rl_1B_2\omega^2/g$$

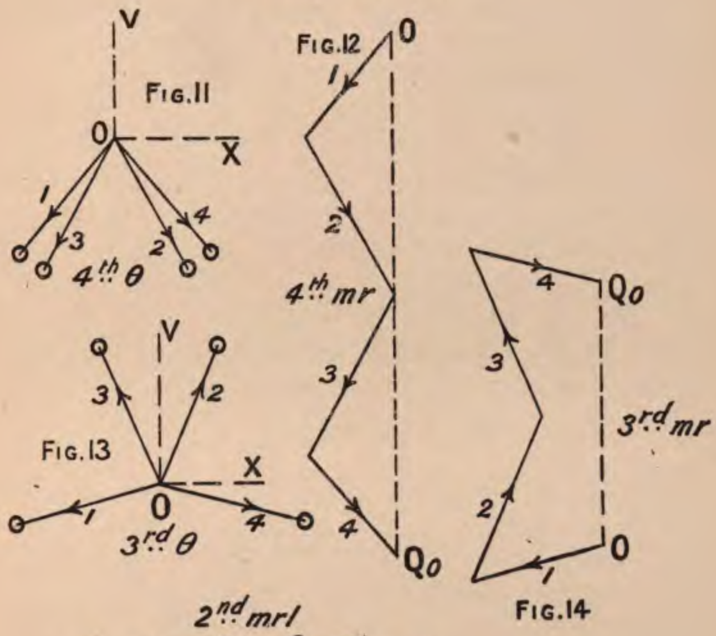

Fig. 11 Fig. 12 Fig. 13 Fig. 14 Fig. 10

(6) **Unbalanced Force of Fourth Order.**—Figs. 11 and 12 are the crank-angle diagram, and mr vector-polygon respectively of the fourth order. Evidently

$$\begin{aligned} OQ_0 &= 2(m_1 \cos 4\theta_1 + m_2 \cos 4\theta_2) \\ &= 2m_1(\cos 4\theta_1 + 2 \cos^2\theta_1 \cos 4\theta_2) \end{aligned}$$

Taking $\theta_1 = 30°$, $\theta_2 = 125° \, 16'$, $m_2 = 1{\cdot}50 m_1$, and OQ_0 is $2m_1(-0{\cdot}500 - 0{\cdot}777 \times 1{\cdot}50) = -3{\cdot}33 m_1$, the maximum value of the unbalanced force of the fourth order is thus

$$3{\cdot}33 m_1 r B_4 \omega^2/g$$

(7) **Transverse Couples.**—The resultant transverse couples of the orders 1, 3, 5, . . . due to the swing of the connecting-rods, are obtained by drawing I vector-polygons as explained in Chap. VII. If the I's of the rods are proportional to the corresponding mr's, the primary I vector-polygon will be closed, and the primary transverse couple is zero. Figs. 13 and 14 are the crank-angle diagram, and I vector-polygon respectively of the third order. Evidently

$$OQ_0 = 2(I_1 \cos 3\theta_1 + I_2 \cos 3\theta_2)$$

Taking $\theta_1 = 30°$, $\theta_2 = 125° \, 16'$, $I_2 = 1{\cdot}50 I_1$, and OQ_0 is $2I_1(0 + 1{\cdot}50 \times 0{\cdot}962) = 2{\cdot}89 I_1$, the maximum value of the couple of the third order is thus—

$$2{\cdot}89 I_1 C_3 \omega^2/g.$$

(8) **Kinetic Energy of Mechanism.**—The average kinetic energy of the pistons is (see Chap. IX.)

$$2(m_1 + m_2) r^2 C_0 \omega^2/2g$$

the average linear energy of the connecting-rods is

$$2(m_{r1} + m_{r2}) r^2 D_0 \omega^2/2g.$$

Taking $\theta_1 = 30°$, these become respectively

$$4{\cdot}98 m_1 r^2 C_0 \omega^2/2g = 2{\cdot}515 P_1$$

and $$4{\cdot}98 m_{r1} r^2 D_0 \omega^2/2g = 3{\cdot}391 L_1$$

where $P_1 = m_1 r^2 \omega^2/2g$ and L_1 is the similar expression for the connecting-rod mass.

The variations of the first and second orders vanish, since the m vector-polygons are closed if the ratio m_2/m_1 for both piston and connecting-rod masses satisfy equation (18). Since the resultant vector of the third order (Fig. 14) is

$2{\cdot}89\ m_1$, the variations of energy of the third order are for the pistons

$$2{\cdot}89 m_1 r^2 C_3 \omega^2/2g = -0{\cdot}293 P_1$$

for the connecting-rods

$$2{\cdot}89 m_{r1} r^2 D_3 \omega^2/2g = -0{\cdot}116 L_1$$

(9) **Uniformity of Torque.**—For a motor-car engine working on the four-stroke cycle, the uneven impulse intervals and irregular torque would more than counterbalance any

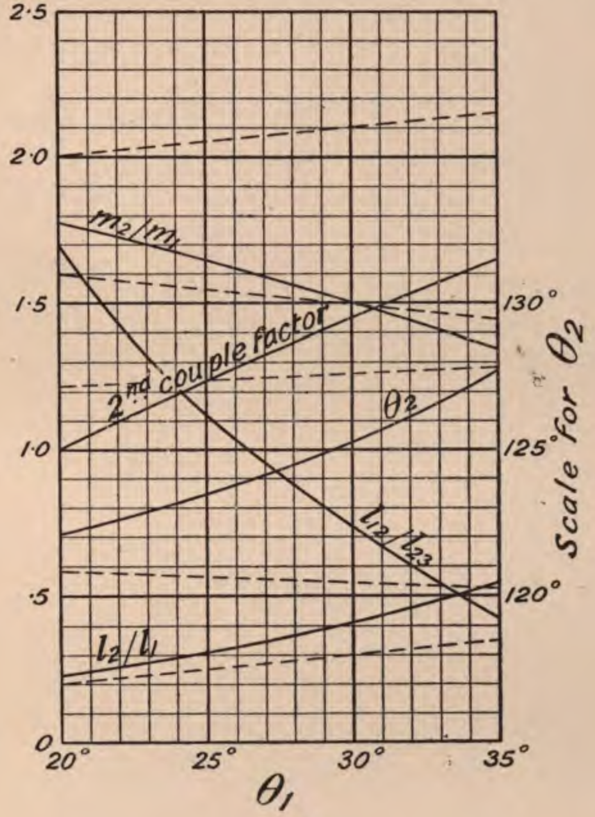

Fig. 15.

advantages of better balance. On the other hand, for a double-acting marine or locomotive steam-engine, the symmetrical four-crank engine affords greater certainty of

TABLE XII.

FOUR-CRANK SYMMETRICAL ENGINES, WITH BEST POSSIBLE BALANCE.

θ_1	θ_2	$\frac{m_2}{m_1}$	$\frac{l_2}{l_1}$	$\frac{l_{12}}{l_{23}}$	Secondary couple factor ($\sin 2\theta_1 + \tan \theta_1$)	Fourth order force factor $2(\cos 4\theta_1 + 2\cos^2\theta_1 \cos 4\theta_2)$	Angles between cranks.
20°	122° 9′	1·77	0·23	1·70	1·007	1·86	40°, 98·9°, 122·2°, 98·9°
22° 30′	122° 46′	1·71	0·27	1·38	1·121	2·24	45°, 96·1°, 122·8°, 96·1°
25°	123° 29′	1·64	0·31	1·12	1·232	2·62	50°, 93·3°, 123·4°, 93·3°
27° 30′	124° 19′	1·57	0·35	0·92	1·340	2·99	55°, 90·3°, 124·4°, 90·3°
30°	125° 16′	1·50	0·41	0·73	1·443	3·32	60°, 87·4°, 125·2°, 87·4°
32° 30′	126° 21′	1·42	0·47	0·57	1·543	3·62	65°, 84·3°, 126·4°, 84·3°
35°	127° 37′	1·34	0·54	0·43	1·640	3·87	70°, 81·2°, 127·6°, 81·2°

starting, and may give a very equable torque. Thus, with $\theta_1 = 20°$, $\theta_2 = 122\frac{1}{4}°$, the cranks pass the dead centres at angular intervals of 40°, $37\frac{3}{4}°$, $64\frac{1}{2}°$, $37\frac{3}{4}°$. If θ_1 is less than 20°, the angular intervals are still more regular, but the ratio of the outside distance l_{12} to the middle distance l_{23}, between the cylinders becomes excessive; the ratio of the masses m_2/m_1 also becomes greater. For equal distances between the cylinders, $l_2/l_1 = 1/3$, and from equation (20), $\cos^2 \theta_1 = 0·800$, $\theta_1 = 26\frac{1}{2}°$, $\theta_2 = 124°$, $m_2 = 1·6\ m_1$, and the cranks pass the dead centres at angular intervals of 53°, $29\frac{1}{2}°$, 68°, $29\frac{1}{2}°$.

The results of the foregoing investigations are given in Table XII., for values of θ_1 covering the useful practical range. They are also exhibited graphically in Fig. 15, the values of θ_1 being taken as the base. The intercepts between the dotted lines are proportional to the angular intervals between the cranks.

(10) **Symmetrical Five-crank Engine with Perfect Primary and Secondary Balance.**—For perfect primary and secondary balance, eight equations similar to (5)–(12) must be satisfied. In the five-crank engine there are $3 \times 5 - 4 = 11$ variables, so three relations may be assumed between the variables.

Consider a symmetrical engine, which is specified by the six relations

$$\begin{aligned} m_1 &= m_5, & m_2 &= m_4 \\ \theta_1 &= -\theta_5, & \theta_2 &= -\theta_4 \\ l_1 &= -l_5, & l_2 &= -l_4 \end{aligned}$$

in which the middle crank, No. 3, is taken as reference line for θ's and l's. Substituting, equations (6), (8), (9) and (11) are identically satisfied, and (5), (7), (10) and (12) become respectively

$$\tfrac{1}{2}m_3 + m_1x_1 + m_2x_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$
$$\tfrac{1}{2}m_3 + m_1(2x_1^2 - 1) + m_2(2x_2^2 - 1) = 0 \quad . \quad (22)$$
$$m_1l_1y_1 + m_2l_2y_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$
$$m_1l_1x_1y_1 + m_2l_2x_2y_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

That is, besides the six assumed relations, the variables must satisfy four equations, leaving one relation to be specified arbitrarily.

From (23) and (24) $$x_1 = x_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Substituting in (21) and (22), and solving for x_1 we get

$$x_1 = 1 \text{ or } -\tfrac{1}{2} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

That is, $\theta_1 = 0°$ or $120°$.

If $x_1 = 1$, then from (21), $m_3 = -2(m_1 + m_2)$, an impracticable solution, since none of the masses can be negative.

If $x_1 = -\frac{1}{2}$, from (21)

$$m_3 = (m_1 + m_2) \quad . \quad . \quad . \quad . \quad . \quad (27)$$

From (25), since $\theta_1 = 120°$, $\theta_2 = 120°$ or $240°$, and $y_1 = \pm y_2$. Then from (23)

$$\frac{l_1}{l_2} = \pm \frac{m_2}{m_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

The angles being all determined, the remaining relation must be taken between the m's and l's; then equations (27) and (28) serve to determine the remaining variables.

The solution thus reduces to the engine with cranks at 120°, shown in Fig. 18, Chap. VI.

CHAPTER XII

DISPLACEMENT OF ENGINE FRAME DUE TO UNBALANCED FORCES AND COUPLES

(1) **Engine bolted down to Foundations.**—The bedplate of an engine is usually bolted down to a structure as rigid and immovable as is consistent with the conditions of the environment. In a stationary land engine, massive foundations of stone, brickwork, or concrete are prepared, and the holding-down bolts should be so arranged as to securely bind the whole of the foundation to the bedplate. The bed-plate will then not move appreciably, even if there be an unbalanced force or couple while the engine is running.

(2) **Rigid Engine Frame on Springs.**—In some cases, however, the engine is bolted to a frame which is more or less elastic, and which may be supported on springs, which allow it a certain freedom of motion. Thus, in a motor-car, the springs and tyres permit the chassis frame to move freely within a limited range, under the action of unbalanced forces or couples of the engine. The locomotive engine also is under somewhat similar but more complex conditions as to suspension, since the crank-shaft of the engine is the same as the driving-wheel axle which rests direct on the rails; while the cylinder and slide-bars are spring supported. The vibratory motion of the engine frame under these conditions is not quite simple; the following short discussion serves to show the nature of the motion.

We will assume that the motion of the engine frame is so small that the forces on the springs do not vary as they

are extended or compressed by the action of the unbalanced force or couple.

Firstly, let the engine have an unbalanced mass reciprocating relative to the engine frame along a straight line of length equal to twice the radius of the crank-pin, and suppose the mass-centre G of the whole engine and frame to coincide with the mean position of this unbalanced mass. That is, the centre of the shaft, and the centre of the mass m will in one position both coincide with G. Let M be the total mass of the engine and frame, including m the unbalanced mass. The mass-centre G remaining fixed in space under the action of the internal force, if the unbalanced mass m reciprocates between the extreme positions m_1 and m_2, the mass-centre F of the remaining mass $(M - m)$ (that is, the centre of the crank-shaft), reciprocates from f_1 to f_2.

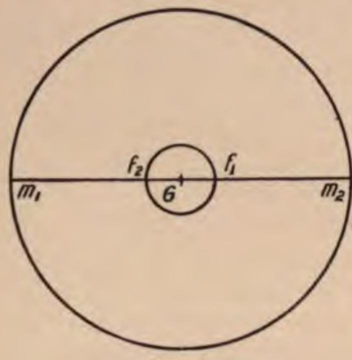

FIG. 1.

also $$m \times Gm_1 = (M - m) \times Gf_1$$

$$\therefore \frac{Gf_1}{Gm_1} = \frac{m}{(M - m)}$$

and therefore

$$\frac{Gf_1}{f_1m_1} = \frac{m}{M}$$

That is, since $f_1m_1 = r$,

$$Gf_1 = mr/M.$$

Therefore, the semi-amplitude of the vibration of the engine frame is m/M times the crank radius.

In the same way, if the unbalanced mass m rotates relative to the axis of the engine shaft, the latter describes a circle in space of radius Gf_1. The small circle (Fig. 1), shows the path of the centre of the shaft, the large circle

shows the path of the crank-pin in space, relative to the mass centre G of the whole engine and frame.

Secondly, suppose, at first, that m reciprocates in the plane of the paper (Fig. 2). When m lies on the axis of the crank-shaft, let the mass-centre G of the whole engine and frame lie on the axis of the shaft at a distance l from the transverse plane of motion of m. The unbalanced force f acts mutually between m and the remaining mass $(M - m)$ of the engine and frame. The mass-centre of the latter is at f_1, and, as before

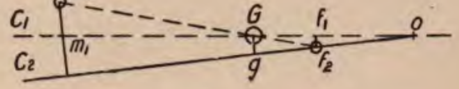

FIG. 2.

$$\frac{Gf_1}{f_1 m_1} = \frac{m}{M}$$

The unbalanced force f on the engine and frame is equivalent to an equal force acting on the mass $(M - m)$ at f_1, and a couple, $f \times m_1 f_1$. The former causes a linear movement of the mass-centre f_1; the latter causes an angular motion of the frame about an axis through the moving mass-centre f_1. The instantaneous position of this axis of rotation depends on the position of the three principal axes of inertia of the engine and frame, and the full discussion of this problem is beyond the scope of the present work. But if the masses of the engine and frame are symmetrically disposed about the plane of the paper, one principal axis of inertia is at right angles to the plane of the paper, and is also the axis of rotation, with the conditions assumed above. Thus, the extreme positions of the unbalanced mass m and the axis of the crank-shaft are shown at m_2 and f_2C_2.

Draw Gg perpendicular to f_2c_2; this measures the displacement of the axis of the shaft from the immovable mass-centre of the system, and, as before, is equal to m/M times the crank radius. Producing C_2f_2 and C_1f_1 to intersect at O, we get a point on the axis of the crank-shaft which has no linear displacement. On the other hand, the linear displacement

of the axis of the shaft at m and further from the mass-centre is increased. The maximum angular displacement θ_1 of the axis of the shaft from its mean position depends on the moment of inertia I_1, of the engine and frame about an axis through G perpendicular to the plane of the paper. Approximately

$$\theta_1 = mrl/I_1$$

If the mass m rotates, the point g describes a circle in space. The uniform circular motion of m may be considered as the sum of two simple harmonic motions. With similar conditions as to the masses being symmetrically disposed about the plane through C_2f_2 perpendicular to the plane of the paper, the maximum angular displacement θ_2 at right angles to the plane of the paper is approximately

$$\theta_2 = mrl/I_2,$$

I_2 being the moment of inertia about the axis Gg. The axis of the crank-shaft then describes, relative to f, a cone of elliptical section.

Thirdly, let the mass-centre G of the whole engine and frame be in the transverse plane of the unbalanced mass m, but at a distance l from the axis of the shaft, m_1G being the line joining G to the mean position of m. Any displacement of m along m_1G will cause a linear displacement of the mass-centre of the frame, as discussed in Fig. 1. Any displacement of m along a line at right angles to m_1G will cause a linear displacement and an angular displacement, as discussed in Fig. 2. If the mass m revolves, a point on the frame describes a circle round G as centre, of radius mr/M. The maximum angular displacement is mrl/I, and takes place about an axis through the mass-centre of the frame parallel to the mean direction of the shaft, I being the moment of inertia about this axis.

Lastly, whatever be the position of the mass-centre G_1 relative to the unbalanced mass m_1, the displacement of a point on the frame whose mean position coincides with G is m/M times that of the unbalanced mass m from its mean position.

(3) **Amplitude of Vibration in Different Types of Engine.**—The unbalanced mass m considered above, may be a part of the actual moving masses of the engine, or it may be an imaginary mass revolving with, or driven by, imaginary cranks of the second, fourth, sixth . . . orders (sect. 4, Chap. VI.). In the Synopsis Chap. XV., the unbalanced *forces* are expressed for the various types of engine. The corresponding displacement of the mass-centre of the engine can be conveniently expressed in terms of the unbalanced force.

In Chap. VI. it has been shown that the maximum values of the piston acceleration forces of the first, second, fourth, sixth . . . orders are equal to $mr\omega^2/g$, multiplied by 1, B_2, B_4, B_6, . . . respectively, and that the corresponding masses driven by the imaginary cranks of radius r are respectively m, $B_2m/4$, $B_4m/16$, $B_6m/36$, . . . respectively. Therefore, for *equal* unbalanced forces of the first, second, fourth, sixth orders, the corresponding masses are in the ratio 1, $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{36}$, . . . , and the displacements of the engine frame produced by these forces are in the same ratio. Consequently, in comparing the amplitudes of engine vibration in different types, the numerical values given in columns 2 and 3, Table XIII., should be divided by 1, 4, 16, 36, . . . according as the unbalanced force or couple is of the first, second, fourth, sixth, . . . order. Similarly, the numerical values given in column 4 should be divided by 1, 9, 25, . . . according as the unbalanced transverse couple is of the first, third, fifth, . . . order.

(4) **Vibration of Elastic Frame.**—If the frame is elastic, as every structure is to a certain degree, it has a certain frequency or periodic time of natural vibration. If the frequency of the disturbances due to unbalanced forces in the engine is the same as the frequency of the natural vibration of the frame, the amplitude of the latter may become excessive, having no direct relation to the amplitude discussed above, considering the whole frame rigid. The case is analogous to the oscillation of a very heavy pendulum. If a small deviating force is applied at regular intervals of

time, a very great swing may be set up if the intervals are the same as, or exact multiples of, the periodic time of swing of the pendulum. For other values of the periodic time of the deviating force, little or no swing may be set up. Thus there is a critical speed of the engine at which the vibration of the elastic frame will be a maximum. If this speed is exceeded, the vibration may become less. Other critical speeds will be 2, 3, 4 . . . times that of the lowest critical speed.

The amplitude of such natural vibrations may be kept low, if some "damping" device can be introduced; that is, some frictional resistance to the natural oscillation. The laminated springs of road and railway vehicles are preferred to other types of springs, mainly on account of the damping action offered by the frictional resistance to sliding of one leaf on the other.

CHAPTER XIII

ENGINES FOR VARIOUS PURPOSES

(1) **Stationary Vertical Engines.**—A vertical engine without balance masses has its unbalanced forces and couples, if any, in a vertical plane. If the bedplate is securely bolted to a foundation of sufficient mass, there is not much to be gained by introducing balance masses. The introduction of balance masses in a vertical engine introduces reciprocating unbalanced forces or couples in the horizontal plane of the crank-shaft, producing horizontal shear on the foundations. In any case, therefore, the proportion of the reciprocating masses balanced by masses on the crank-shaft should be less than one-half. When the engine is half balanced, the vertical disturbing effect is reduced to one-half, and the horizontal and vertical disturbing forces are equal.

(2) **Stationary Horizontal Gas- and Steam-engines.**—The same considerations, applied to the horizontal engine, would indicate that the proportion of the reciprocating masses balanced by masses on the crank-shaft should be greater than one-half. The reciprocating masses may be even completely balanced, thus eliminating the primary horizontal forces and introducing vertical unbalanced forces.

In a small gas-engine, in which the cylinder is horizontal and overhung beyond the bedplate, the foundation does not extend the full length of the engine. Consequently, a horizontal unbalanced force will usually have a greater disturbing effect than an equal vertical force. The counterbalance mass may therefore be relatively larger than in an engine with a longer foundation.

In a tandem engine, either gas or steam, with either one or two cranks, the reciprocating masses may be half-balanced.

(3) **Locomotives.**—The design of the locomotive engine is influenced chiefly by the position of the driving-wheels, which cannot be altered. In a two-cylinder locomotive, the cylinders are placed either "inside" or "outside" the frame. For convenience of starting and reversing, the cranks are at right angles, consequently there is an unbalanced force and couple in a horizontal plane which has a tendency to derail the locomotive. The magnitude of this couple is proportional to the distance between the centre lines of the cylinders; it is therefore much greater with "outside" than with "inside" cylinders. To minimize this derailing couple, counter-balance masses are usually fixed on the driving-wheels, which have the effect of introducing a vertical unbalanced force and couple, producing the "hammer blow" on the rail. (See sect. 6, Chap. V.)

If the vertical and horizontal forces and couples were equally objectionable, then probably the best result would be got by half-balancing. If the derailing couple is required to be completely eliminated, the balance masses should correspond to the total mass of the reciprocating parts. But in this matter the locomotive engineer has to be guided by experience. If the balance masses are too great, the hammer-blow at one position of the crank-shaft is excessive, and at another position the driving-wheel tends to lift from the rail. The better and heavier the permanent way, probably the greater is the proportion of the reciprocating parts that can be balanced with all-round advantage.

As regards the derailing couple, this must be reduced to within safe limits, whatever be the vertical blows introduced. The "outside" cylinder engine should therefore have relatively greater balance masses than the "inside."

The resultant unbalanced horizontal *force* in a two-cylinder locomotive is less objectionable than the vertical force or the vertical and horizontal couples; it merely produces a cyclic variation of the pull on the draw-bar.

With four- and six-wheel coupled locomotives, the counterbalance masses can be divided between the driving-wheels, thus distributing the hammer-blow, and being less severe on the permanent way. It thus appears that in coupled engines greater balance masses can be used with all-round advantage than in an engine with only one pair of driving-wheels.

Thus, the counterbalance masses should be relatively smallest in the single-drivers inside-cylinders locomotive, relatively largest in the outside-cylinders coupled engine (see Tables III. and IV., Chap. V.). In the best practice, from one-third to three-fourths of the reciprocating masses are balanced, the proportion depending on the foregoing considerations.

It should be noted that the crank-shaft bearings of a locomotive are mounted on the horn-plates and supported by the springs from the frame. Thus the engine frame (including cylinders, guide-bars, and crank-shaft bearings) is not a rigid structure, as is the case with nearly all other engines. A thrust or pull along the connecting-rod, whether due to steam pressure or inertia forces, tends to displace the springs, except when the crank is on the dead centre, or the rod horizontal. The action of the springs may therefore slightly modify the mutual action between rails, wheels, and frame; but the deductions above are sufficiently accurate for the purpose of comparing different types of locomotives.

(4) **Four-cylinder Compound Locomotive.**—A four-crank locomotive engine, designed as discussed in Chap. XI., has its primary balance perfect. The secondary couple (see Synopsis No. 22) is still appreciably large, being from one-fourth to one-third the primary couple due to an engine with two cranks at 180°, and with outside cylinders. This is a derailing couple, alternating from side to side, but as the periodic time of a complete cycle is half that of a revolution of the driving-wheels, its effect in producing displacement of the locomotive is one-fourth that of a primary couple of the same magnitude (see sect. 4, Chap. VI.). If the cylinders are exactly horizontal, there is no hammer-blow on the rails,

since the couple lies in the horizontal plane. The variation of pull on the draw-bar due to unbalanced forces, being of the fourth order, is negligibly small.

(5) **Marine Engines.**—The most general type is double acting with vertical cylinders, the steam being expanded in two, three, or four cylinders successively. For convenience of easy starting and reversing, two cranks at least are necessary; three cranks are still better. The compound engine has usually two cranks at right angles, this giving the most uniform torque consistent with the number of cylinders. Counterbalance masses are not usually fitted. The unbalanced force and couples are shown in No. 5 Synopsis at end of book, and act in the vertical plane of the cylinders. The action is therefore alternately up and down on the floor of the ship. The unbalanced force and longitudinal couple can be reduced to half their maximum values by counterbalance masses (No. 6 Synopsis). They are still very large, and for comparative freedom from vibration, two cranks are not sufficient. The engine with three cylinders, and three cranks at 120°, gives a very uniform torque, and if the reciprocating masses are equal, there is no appreciable unbalanced force (Nos. 15 and 16 Synopsis), only a primary longitudinal couple.

Four-crank Engine.—Presuming the eight torque curves from each of the ends of the cylinders are identical, the most uniform torque on the crank-shaft would be got with the crank angles 90°, 45°, 90°, 135°. The cranks would then pass the dead centres at equal angular intervals of 45°. The balance, however, with equal reciprocating masses would not be the best possible with four cranks (Nos. 20 and 21).

The four-crank symmetrical engine discussed in Chap. XI. is specially suitable for marine engines, the balance being the best possible, and the torque may be made fairly uniform.

(6) **Motor-car Engines.**—The balance of the various types of engines used on motor-cars has been exhaustively discussed in the previous chapters, and the principal results are conveniently shown in the Synopsis. Here, only a few general remarks are required.

It must be kept clearly in view that the balance of an engine is only one factor in the general problem of its design, and other factors may be of greater importance. Uniformity of torque, to a certain degree, is desirable, hence the general use of four-cylinder engines (No. 19 Synopsis) on cars of moderate power. For a still greater degree of uniformity of torque, six cylinders (No. 31) have long been used on the Napier cars, and many other makers appear to be following this example. The torque of the eight-cylinder engine (No. 34 Synopsis) is still more nearly uniform, but the balance is not so good as with the six-cylinder engine. This fact, combined with the greater multiplicity of parts, has prevented the extended use of the eight-cylinder engine, except on racing cars of abnormal power.

The five-cylinder engines (No. 28 and 29) are types that may possibly be used, but the author is not aware that either has been used up to the present. In uniformity of torque they are intermediate between the four-cylinder and the six-cylinder engine. In balance they are inferior to the six-cylinder. No. 29 is superior to the four-cylinder, probably so also is No. 28; in the latter *the largest disturbance being a small primary couple,* in the four-cylinder a secondary force. In respect of the flywheel effect of the mechanism (uniformity of kinetic energy of pistons and connecting-rods) the five-cylinder engine is immensely superior to the six-cylinder, the latter superior to the four-cylinder, as the figures in columns 7 and 8 of the Synopsis show. Probably the pistons and connecting-rods of the five-cylinder engine can be made with advantage more massive than those of a four- or six-cylinder engine, and that with no appreciable flywheel.

In engines with less than four cylinders, the torque is less uniform as the number of cylinders is less. As regards balance, the three-cylinder engine (No. 16) may, in some circumstances, be quite as satisfactory as the four-cylinder. When the engine is placed, in the usual position, in the front of the chassis, an unbalanced force in the engine is equivalent to a force and a couple on the chassis (see Chap.

XII.). The displacement of certain parts of the chassis due to engine vibration may thus be greater with a four-cylinder engine than with a three-cylinder.

Of the two-cylinder two-crank engines, those with the cranks at 180° are the better balanced, but the impulses come at angular intervals of $\frac{1}{2}$ and $1\frac{1}{2}$ revolutions. The general practice seems to put uniformity of torque as of greater importance than good balance, most of the two-cylinder engines having their cranks in line, and the impulses at regular intervals of one revolution; the balance is then equivalent to that of a single-cylinder engine.

The two-cylinder V-type engine with one crank, cylinders at 90° (No. 12), is better balanced than the four-cylinder engine (No. 19), and is immensely superior, in this respect, to the two-crank two-cylinder engines. The impulses are at intervals of $\frac{3}{4}$ and $1\frac{1}{4}$ revolution, the deviation from equality of intervals not being serious. The factory cost of this engine should be less than that of a two-crank engine of equal power, if manufactured on a sufficient scale.

Two-cylinder V-type engines with cylinders at angles other than 90° are used largely for motor-bicycles and tricars, and various combinations of V-type engines with two or more cranks are possible. Some of these are shown in the Synopsis. In many of these possible combinations, either the impulses are not at regular intervals or the balance is not very good. From this double point of view the eight-cylinder four-crank engines (Nos. 34 and 35) are the most satisfactory combinations. No. 34, with the four cranks all in the same longitudinal plane, is the type usually made for high-power cars and for motor-boats; but No. 35, with cranks at right angles, is much better balanced, the unbalanced force being one-hundredth part, and the displacement of the engine due to want of balance being one four-hundredth part of the respective amounts in No. 34.

One-crank three-cylinder engines with cylinders at about 60° (No. 18), have been used to a limited extent, but have not made much progress. The balance is not very good, and

the impulses are irregular. The three-cylinder one-crank engine, with cylinders at 120° (No. 17), has been used for many years as a steam-engine. Used as a gas or petrol engine, working on a four-stroke cycle, the impulses are at regular intervals of two-thirds of a revolution. The balance is better than that of the four-cylinder engine (No. 19).

The four-cylinder one-crank engine (No. 23) is in mathematically perfect balance, but, working on a four-stroke cycle, the impulses would be at intervals of $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ of a revolution.

In the five-cylinder one-crank engine (No. 30) the impulses are at regular intervals of two-fifths of a revolution, and the balance is so good as to be considered practically perfect. The flywheel effect of the pistons and connecting-rods is also practically perfect. If used on a motor-car, the longitudinal space occupied by the engine is small, so a large roomy body can be used without the necessity of a very long wheel-base. The diametral space occupied (about seven or eight times the stroke) would probably tend to the use of a short stroke (as compared with diameter of cylinder), and the arrangement of the transmission gear might have to be modified somewhat.

In the radial-cylinder engines above discussed, it may at first be thought that lubrication difficulties may arise from oil in the crank-chamber flowing only to the lower cylinders. But even at moderate engine speeds the piston acceleration is much greater than that of a body falling freely under the influence of gravity. In an engine of 4-inch stroke, at a speed of 1000 revolutions per minute, the piston acceleration is nearly fifty-seven times that due to gravity. The oil would thus be thrown violently from the insides of the pistons.

The single-cylinder engine up to eight or ten H.-P., with a large flywheel, gives fairly satisfactory results. The torque is uneven, and at low speeds the unevenness of driving is marked. At higher speeds the unevenness is not so noticeable. The engine should be half balanced. The

effect of the flywheel of a motor-car engine is discussed in Chap. X. Some makers use single-cylinder engines on their cars of lowest price, and four-cylinder engines on cars of higher price; their opinions evidently being that it is not worth while manufacturing an engine intermediate between these two types; the two- and three-cylinder types being defective either in balance or evenness of torque as compared with the four-cylinder.

Many other types of engines have been used on motor-cars; the figures as to balance of the more important types are given in the Synopsis.

(7) **Motor-cycle Engines.**—The single-cylinder and two-cylinder one-crank types are the most frequently used for motor bicycles and tricars. In the V-type engine, the cylinders are often placed at an angle of from 45° to 60°, presumably because such an arrangement does not demand so much space as when the cylinders are at 90°. With an angle of 60° (No. 11), the balance is much worse than with 90°; the unbalanced forces in the two cases being in the ratio 24·67 : 8·14. But as regards the vibration, or displacement, of the engine due to imperfect balance, in the 90° V-type engine the unbalanced force is of the second order, and its relative figure must be divided by 4. That is, the amplitudes of the vibration in the two cases will be in the ratio 24·67 : 2·03, or approximately 12 : 1. It seems worth while providing the extra space to produce this desirable result.

(8) **Petrol Engines for Air-ships.**—If the aëroplane or air-ship, depending not on buoyancy of gases but on engine power for successful flight, is ever manufactured commercially (as seems quite probable, since all the elements for success are to hand, and only require skilful co-ordination), the radial-cylinder type of engine may well become most prominent. Already M. Santos Dumont has constructed an aëroplane which has flown a certain distance, using an eight-cylinder four-crank engine (No. 34). The five-cylinder one-crank engine (No. 30) can probably be made lighter for a given horse-power than any of the types hitherto used; and

its other mechanical properties are very good. For larger powers, a radial six-cylinder two-crank engine (No. 32) and a ten-cylinder two-crank engine (No. 37) are possible types. In both these engines the cranks are close together, just allowing sufficient distance for the crank-cheek to join the two crank-pins, without an intermediate bearing.

CHAPTER XIV

LARGE GAS-ENGINES

(1) Gas-engines for Marine Purposes.—Large gas-engines developing 1000 H.-P. and upwards in one cylinder have been made, principally to utilize the waste gases from the blast furnaces of ironworks. Many of these large engines are double-acting, and work on a two-stroke cycle, so that an impulse is given at each stroke. The description of this or any other type of engine is beyond the scope of the present work, which only deals with the inertia forces. The steady development in suction gas-producers is beginning to direct attention to the subject of gas-engines for marine purposes. The following notes, which are purely speculative, may be of interest.

The inertia forces in a large one-cylinder gas-engine are so great that massive foundations are necessary. On board ship, massive foundations are inadmissible, and the engine is fastened direct to the floor beams forming an integral part of the hull. The hull being an elastic structure, any unbalanced force from the engine sets up vibration, and if the frequency of the disturbing force coincides with the frequency of the natural vibrations of the hull, the amplitude of the latter may become excessive. Hence the necessity for a well-balanced engine.

Leaving aside the question of the mechanism of starting, reversing, etc., in order to start, stop, reverse a large marine engine quickly, it should have no flywheel; or, at least, the greater the flywheel effect the slower will be the manœuvring. This condition would debar the use of the four-cylinder

engine with cranks at 180°, since all the piston masses come to rest when the cranks are on the dead centre. A fairly constant torque on the shaft is also essential; this, again, is against the four-cylinder engine, working on the usual four-stroke cycle. A four-crank engine, with cranks at 90°, and either double-acting cylinders, or tandem cylinders single acting, would be better. As regards uniformity of torque, there would be four impulses per revolution, as in a two-cylinder double-acting steam-engine with cranks at 90°. The flywheel effect of the mechanism would be good, the cyclic variation of the crank-shaft speed from this cause (assuming no flywheel effect from the crank-shaft and propeller) would be about 0·2 per cent. above and below the mean. The eight-cylinder four-crank engine with cranks at 180°, and pairs of cylinders at 90° for each crank, is similar in its dynamical properties (see No. 34 Synopsis at end of the book). The similar engine, No. 35, but with cranks at 90°, is considerably better balanced, the unbalanced force being one-hundredth part that of No. 34. The disturbing force in No. 34 would be parallel to the floor of the ship, in which direction the stiffness is greater than in the vertical direction.

The eight-cylinder four-crank petrol engine, No. 34, has already been extensively used for launches. For gas-engines of great power, say 1000 H.-P. and upwards, to be used on board ship, eight-cylinders, if single acting, using the four-stroke cycle, would seem to be the smallest number consistent with steady torque, good balance, and steady running without a flywheel.

(2) **Possible New Types.**—For still greater powers, increasing the number of cylinders, instead of increasing their size, merits consideration. For this reason the dynamical properties of various engines with from ten to twenty-four cylinders have been tabulated in the Synopsis, Nos. 37–43. The four-cylinder one-crank engine, with cylinders at 90° (No. 23), is perfectly balanced. With the four-stroke cycle, impulses in this engine cannot be at equal intervals, but two sets together with cranks either in line, at 90° or 180° (No. 36), will give impulses at equal intervals. Two sets of No. 36,

with cranks at 45°, will give an engine with sixteen cylinders and four cranks (No. 42). In the eight-cylinder engine, the torque on the shaft due to the pressure of gases in the cylinders varies appreciably (Fig. 5, Chap. X.), but as the curve does not very appreciably deviate from a sine curve, if two be superposed, the resultant will not deviate much from a straight line parallel to the base. Presuming that the indicator diagrams in all the cylinders are identical, this engine, therefore, at all speeds and powers, should give practically uniform speed and torque. Three sets of No. 36 give a twenty-four-cylinder six-crank engine (No. 43). These two engines, Nos. 42 and 43, being perfectly balanced, having practically constant torque, and their moving parts constant kinetic energy for uniform speed of the shaft, should compare with the steam-turbine in smoothness of running and absence of vibration.

Particulars as to three types of engine each with twelve cylinders are given, Nos. 38, 39, 40. Neither of these seems to be so satisfactory as either the eight- or sixteen-cylinder engines, Nos. 34, 35, and 42, either a large unbalanced force or irregularity of impulse being a drawback. No. 41, with twelve cylinders, six cranks, formed from six sets of No. 11, is quite satisfactory.

The cubic engine-room space occupied by the various multicylinder engine formed by sets of four cylinders at 90° with one crank, should compare favourably with that for any other type of engine of equal power. With ratio of connecting-rod to crank length four or five, the extreme diameter over cylinder ends should not be more than seven or eight times the stroke.

(3) **Small Units.**—Single-acting four-stroke cycle gas-engines giving 100–200 H.-P. per cylinder, have for years past been running successfully. The combination of a large number of these units to give an engine of great power, should present less difficulties than an attempt to get more power from each cylinder. The cost of production should be less, since cylinders, pistons, rods, could be each of one or two standard sizes, say 100 H.-P. and 250 H.-P. With 8,

12, 16, and 24-cylinder engines built up from these units, engines of 800, 1200, 1600, 2000, 2400, 3000, 4000, and 6000 H.-P. could quickly be turned out by the engine maker.

(4) **Gas-engines for Electric Power Stations.**—The same arguments would apply to the application of large gas-engines for generation of electrical energy. One uniform size of cylinder being used in a power station, grouped in units of various numbers of cylinders, so as to suit the variations of load, the advantages in the form of low initial cost, repairs and renewals, facility of extending the output of the station, small floor space occupied, would be very great.

CHAPTER XV

SYNOPSIS OF RESULTS

In the columns of Table XIII. are shown some of the principal results obtained in the preceding discussions. The first column specifies the type of the engine. For each engine with two or more cylinders the piston masses are assumed each equal; the connecting-rods, also, are assumed equal and similar as regards mass, mass-centre, and moment of inertia. The second column gives the largest unbalanced force (that is, of lowest order), its order being stated in the first line. In this line is also indicated the nature of this unbalanced force. If it is of constant magnitude rotating with, or opposite to, the crank, the words *forward* or *reverse* are used. If the unbalanced force is always in the same direction, like that due to a mass reciprocating along a straight line, the word "parallel" indicates that the direction is parallel to the axes of the cylinders if the latter all lie in the same longitudinal plane, or parallel to an axis of symmetry if the cylinders lie in different longitudinal planes. The word "perpendicular" indicates that the direction is perpendicular to the longitudinal plane of symmetry. If the unbalanced force varies both in magnitude and direction, the word "elliptical" is used. The second line gives the general approximate expression for the largest unbalanced force, the symbol A being used for the expression $mr\omega^2/g$. The third line gives its exact value in terms of the coefficients, Table V., Chap. VI. The fourth line gives its exact numerical value corresponding to the data given below.

Column 3 gives the largest unbalanced couple, the four lines for each engine being as in column 2. Column 4 gives the largest transverse couple due to the angular swing of the connecting-rods, the symbol B being used for the expression $I\omega^2/g$. The third line gives its exact value in terms of the coefficients, Table VI., Chap. VII., the fourth line its numerical value for the assumed data.

Columns 5 and 6 give respectively the average kinetic energy of the pistons and the linear kinetic energy of connecting-rods. To make the information complete, another column should give the average angular kinetic energy of the connecting-rods; but as this is small compared with the items in columns 5 and 6, it has been omitted.

Columns 7 and 8 give the largest variations of kinetic energy. The variations of the first and third orders in any engine where neither of them cancels out, being of about equal magnitude (see Tables VII. and VIII., Chap. IX.), they are both specified. Their algebraic sum is equal to their arithmetical sum for one crank position, to their arithmetical difference for a crank position differing by 180°. Also that of the second order is larger than that of the first, and is specified if it does not cancel out. The variations of higher order than the third rapidly decrease in magnitude as the order is higher; therefore, if a variation of either of the orders one, two, or three is specified, those of higher order are omitted.

As the coefficients C_0, C_1, C_2 involved in the energy of the pistons do not vary greatly for different values of $q = r/l$, particularly those of lower orders, in columns 5–8 are given the values corresponding to $q = 1/5$. Similarly, for the coefficients D_0, D_1, D_2, . . . involved in the linear energy of the connecting-rods; here the mass-centre of the rod has been assumed at two-fifths of the length from the big end.

A comparison of columns 5–8 gives a quick indication of the flywheel effect of the mechanism.

In column 9 is given an approximate estimate of the value of the energy ratio discussed in Chap. X.; that is,

the ratio of the excess energy developed by the engine at any part of the revolution, to the average energy per revolution, on the assumption that the efforts from the cylinders are approximately equal and similar. The ratio is, of course, different according as the engine is used as a four-stroke cycle gas-engine, a single-acting steam-engine, or a double-acting steam-engine. In some cases, therefore, two values are tabulated. For eight cylinders and upwards, only the values for the four-stroke cycle engine are tabulated.

Table XIII. thus gives a complete synopsis of the principal mechanical properties of each engine.

Data for Numerical Values.—To enable the tabulated numerical values to serve as bases from which the corresponding values for any engine with any given data may be quickly calculated, the assumed data is all in powers of ten of the usual British engineering units. Let speed of crank-shaft = 100 revs. per minute, then

$\omega = 10{\cdot}47$, and $\omega^2 = 109{\cdot}4$,
crank radius $= r = 10$ inches $= 0{\cdot}8333$ ft.,
mass of each piston $= m_p = 10$ lbs.,
mass of each connecting-rod $= m_r = 10$ lbs.,
moment of inertia of connecting-rod $= \mathrm{I} = 20$ lb.-ft.^{2},
ratio of crank to connecting-rod $= q = 1/5$,
distance between cylinders $= l = 10$ in. $= 0{\cdot}8333$ ft.

Then unbalanced force of mass m at crank-pin is

$$\mathrm{A} = mr\omega^2/g = \frac{10 \times 109{\cdot}4 \times 0{\cdot}8333}{32{\cdot}2} = 28{\cdot}49 \text{ lbs. weight.}$$

Unbalanced couple of two masses m at crank-pins, cranks at 180°, is

$$\mathrm{A}l = 28{\cdot}49 \times 0{\cdot}8333 = 23{\cdot}78 \text{ lb.-ft.}$$

Let $$\mathrm{B} = \mathrm{I}\omega^2/g = \frac{20 \times 109{\cdot}4}{32{\cdot}2} = 68{\cdot}00 \text{ lb.-ft.}$$

The maximum value of the primary transverse couple is

$$\mathrm{C}_1\mathrm{B} = 0{\cdot}2010 \times 68{\cdot}00 = 13{\cdot}66 \text{ lb.-ft.}$$

The kinetic energy of the mass m_p at crank-pin is

$$P = m_p r^2 \omega^2/2g = \frac{10 \times 109{\cdot}4 \times 0{\cdot}6944}{2 \times 32{\cdot}2} = 11{\cdot}79 \text{ ft.-lbs.}$$

Similarly $\qquad L = m_r r^2 \omega^2/2g = 11{\cdot}79$ ft.-lbs.

TABLE

Synopsis

	1	2	3	4
		Largest unbalanced		
No.	**Type of Engine.**	**Force.**	**Longitudinal couple.**	**Transverse couple.**
		One Cylinder.		
1	No balance masses	1st parallel A .... 28·49 lbs.	None	
2	Half-balanced	1st reverse $\frac{1}{2}$A .... 14·24 lbs.	None	1st Bq C_1B 13·66 lb.-ft.
3	Over-balanced	1st perpendr. A .... 28·49 lbs.	None	
4	Connecting-rod balanced $q=\frac{1}{4}$, $a=\frac{6}{5}$, $b=-\frac{1}{5}$ (Fig. 1, Chap. V.)	None	None	1st $B'q$ C_1B' 136·6 lb.-ft.

XIII.

OF RESULTS.

5	6	7	8	9
Average kinetic energy of		Largest variation of kinetic energy of		Energy ratio, $\frac{\Delta E}{R}$
Pistons.	Connecting-rods, linear.	Pistons.	Connecting-rods, linear.	
		ONE CYLINDER.		
0·505P 5·95 ft.-lbs.	0·681L 8·02 ft.-lbs.	1st, 0·101P 1·19 ft.-lbs. 2nd, −0·500P −5·89 ft.-lbs. 3rd, −0·102P −1·20 ft.-lbs.	1st, 0·040L 0·47 ft.-lb. 2nd, −0·350L −3·77 ft.-lbs. 3rd, −0·041L −0·48 ft.-lb.	4-stroke cycle engines, 1·6–1·8 double-acting steam-engines, 0·20–0·30
	1·220L[1] 9·78 ft.-lbs.		1st, −0·025L[1] −0·20 ft.-lb. 2nd, 0·220L[1] 1·76 ft.-lbs. 3rd, 0·026L[1] 0·21 ft.-lb.	

<table>
<tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><th rowspan="2">No.</th><th rowspan="2">Type of Engine.</th><th colspan="3">Largest unbalanced</th></tr>
<tr><th>Force.</th><th>Longitudinal couple.</th><th>Transverse couple.</th></tr>
<tr><td colspan="5">Two Cylinders, Two Cranks.</td></tr>
<tr><td>5</td><td>Cranks at 90°
No balance masses</td><td>1st parallel
$\sqrt{2}A$
....
40·28 lbs.</td><td>1st parallel
$\frac{1}{\sqrt{2}}Al$
....
16·81 lb.-ft.</td><td rowspan="2">1st
$\sqrt{2}Bq$
$\sqrt{2}C_1B$
19·31 lb.-ft.</td></tr>
<tr><td>6</td><td>Cranks at 90°
Locomotive
Half-balanced</td><td>1st reverse
$\frac{1}{\sqrt{2}}A$
....
20·14 lbs.</td><td>1st reverse
$\frac{1}{2\sqrt{2}}Al$
....
8·40 lb.-ft.</td></tr>
<tr><td>7</td><td>Cranks at 180°
No balance masses</td><td rowspan="2">2nd parallel
$2Aq$
$2B_2A$
11·51 lbs.</td><td>1st parallel
Al
....
23·78 lb.-ft.</td><td>None</td></tr>
<tr><td>8</td><td>Cranks at 180°
Half-balanced</td><td>1st reverse
$\frac{1}{2}Al$
....
11·89 lb.-ft.</td><td>None</td></tr>
<tr><td>9</td><td>Cranks at 180°
Connecting-rods balanced
2 Sets of No. 4
(Fig. 1, Chap. V.)</td><td>None</td><td>None</td><td>None</td></tr>
<tr><td>10</td><td>Opposed cylinders
Cranks at 180°</td><td>None</td><td>1st parallel
Al
....
23·78 lb.-ft.</td><td>1st
$2B$
$2C_1B$
27·32 lb.-ft.</td></tr>
</table>

5	6	7	8	9
Average kinetic energy of		Largest variation of kinetic energy of		Energy ratio, $\frac{\Delta E}{R}$
Pistons.	Connecting-rods, linear.	Pistons.	Connecting-rods, linear.	

Two Cylinders, Two Cranks.

1·010P 1·91 ft.-lbs.	1·362L 16·06 ft.-lbs.	1st, 0·143P 1·68 ft.-lbs. 3rd, −0·144P −1·69 ft.-lbs.	1st, 0·057L 0·66 ft.-lb. 3rd, −0·057L −0·66 ft.-lb.	double-acting steam-engine, 0·02–0·04
		2nd, − 1·000P −11·79 ft.-lbs.	2nd, −0·640L −7·41 ft.-lbs.	4-stroke cycle engine, 1·4–1·6 single-acting steam-engine, 0·15–0·25
	2·440L[1] 19·56 ft.-lbs.		2nd, 0·440L[1] 3·52 ft.-lbs.	
	1·362L 16·06 ft.-lbs.	1st, 0·202P 2·38 ft.-lbs. 2nd, − 1·000P −11·79 ft.-lbs. 3rd, − 0·204P − 2·41 ft.-lbs.	1st, 0·080L 0·94 ft.-lb. 2nd, −0·640L −7·55 ft.-lbs. 3rd, −0·082L −0·97 ft.-lb.	4-stroke cycle engine or single-acting steam-engine 0·80–1·10

<table>
<tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><th rowspan="2">No.</th><th rowspan="2">Type of Engine.</th><th colspan="3">Largest unbalanced</th></tr>
<tr><th>Force.</th><th>Longitudinal couple.</th><th>Transverse couple.</th></tr>
<tr><td colspan="5">TWO CYLINDERS, ONE CRANK.</td></tr>
<tr><td>11</td><td>Cylinders at 60°
Half-balanced
Impulse intervals 300°
420°</td><td>1st reverse
$\frac{\sqrt{3}}{2}$A
....
24·67 lbs.</td><td>None</td><td>1st
$\sqrt{3}$Bq
$\sqrt{3}$C$_1$B
23·66 lb.-ft.</td></tr>
<tr><td>12</td><td>Cylinders at 90°
Half-balanced
Impulse intervals 270°
450°
(Fig. 7, Chap. VIII.)</td><td>2nd perpendr.
$\sqrt{2}$Aq
$\sqrt{2}$B$_2$A
8·14 lbs.</td><td>None</td><td>1st
$\sqrt{2}$Bq
$\sqrt{2}$C$_1$B
19·31 lb.-ft.</td></tr>
<tr><td>13</td><td>Cylinders at 120°
Half-balanced
Impulse intervals 240°
480°</td><td>1st reverse
$\frac{1}{2}$A
....
14·24 lbs.</td><td>None</td><td>1st
Bq
C$_1$B
13·66 lb.-ft.</td></tr>
<tr><td>14</td><td>Cylinders at 180°
Half-balanced
Impulse intervals 180°
540°</td><td>1st reverse
A
....
28·49 lbs.</td><td>None</td><td>None</td></tr>
<tr><td colspan="5">THREE CYLINDERS.</td></tr>
<tr><td>15</td><td>3 Cranks at 120°,
No balance masses</td><td rowspan="2">6th parallel
$\frac{27}{128}$ Aq^5
3B$_6$A
0·001b.</td><td>1st parallel
$\sqrt{3}$Al
....
41·19 lb.-ft.</td><td rowspan="3">3rd
$\frac{9}{8}$Bq^3
3C$_3$B
0·61 lb.-ft.</td></tr>
<tr><td>16</td><td>3 Cranks at 120°
Half-balanced</td><td>1st reverse
$\frac{\sqrt{3}}{2}$Al
....
20·59 lb.-ft.</td></tr>
<tr><td>17</td><td>1 Crank
Cylinders at 120°
Half-balanced</td><td>2nd reverse
$\frac{3}{2}$Aq
$\frac{3}{2}$B$_2$A
8·63 lbs.</td><td>None</td></tr>
<tr><td>18</td><td>1 Crank
Cylinders at 60°
Half-balanced
Impulse intervals 120°, 300°, 300°</td><td>2nd elliptical
$(1 \mp \frac{1}{2})$ Aq
$(1 \mp \frac{1}{2})$ B$_2$A
8·63 to 2·88 lbs.</td><td>None</td><td>1st
2Bq
2C$_1$B
27·32 lb.-ft.</td></tr>
</table>

<table>
<tr><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th></tr>
<tr><th colspan="2">Average kinetic energy of</th><th colspan="2">Largest variation of kinetic energy of</th><th rowspan="2">Energy ratio, $\frac{\Delta E}{R}$</th></tr>
<tr><th>Pistons.</th><th>Connecting-rods, linear.</th><th>Pistons.</th><th>Connecting-rods, linear.</th></tr>
<tr><td colspan="5">Two Cylinders, One Crank.</td></tr>
<tr><td rowspan="4">1·010P
11·91 ft.-lbs.</td><td rowspan="4">1·362L
16·06 ft.-lbs.</td><td>1st, 0·175P
2·02 ft.-lbs.
2nd, −0·500P
−5·89 ft.-lbs.</td><td>1st, 0·069L
0·81 ft.-lb.
2nd, −0·320L
−3·77 ft.-lbs.</td><td>4-stroke cycle engine,
0·90–1·20</td></tr>
<tr><td>1st, 0·143P
1·68 ft.-lbs.
3rd, −0·144P
−1·69 ft.-lbs.</td><td>1st, 0·057L
0·66 ft.-lb.
3rd, −0·057L
−0·66 ft.-lb.</td><td>4-stroke cycle engine,
1·06–1·30</td></tr>
<tr><td>1st, 0·101P
1·19 ft.-lbs.
2nd, −0·500P
−5·89 ft.-lbs.
3rd, −0·202P
−2·34 ft.-lbs.</td><td>1st, 0·040L
0·47 ft.-lb.
2nd, −0·320L
−3·77 ft.-lbs.
3rd, −0·081L
−0·93 ft.-lb.</td><td>4-stroke cycle engine,
1·20–1·45</td></tr>
<tr><td>2nd, −1·000P
−11·59 ft.-lbs.</td><td>2nd, −0·640L
−7·41 ft.-lbs.</td><td>4-stroke cycle engine,
1·4–1·6</td></tr>
<tr><td colspan="5">Three Cylinders.</td></tr>
<tr><td rowspan="2">1·515P
17·86 ft.-lbs.</td><td rowspan="2">2·042L
24·07 ft.-lbs.</td><td>3rd, −0·305P
−3·59 ft.-lbs.</td><td>3rd, −0·122L
−1·43 ft.-lbs.</td><td>4-stroke cycle engine,
0·25–0·40;
double-acting steam-engine,
0·01–0·02</td></tr>
<tr><td>1st, 0·143P
1·68 ft.-lbs.
3rd, −0·144P
−1·69 ft.-lbs.</td><td>1st, 0·057L
0·66 ft.-lb.
3rd, −0·057L
−0·66 ft.-lb.</td><td>4-stroke cycle engine,
0·10–0·16;
double-acting steam-engine,
0·01–0·02</td></tr>
</table>

<table>
<tr><th></th><th>1</th><th>2</th><th>3</th><th>4</th></tr>
<tr><th rowspan="2">No</th><th rowspan="2">Type of Engine.</th><th colspan="3">Largest unbalanced</th></tr>
<tr><th>Force.</th><th>Longitudinal couple.</th><th>Transverse couple.</th></tr>
<tr><td colspan="5">Four Cylinders.</td></tr>
<tr><td>19</td><td>4 Cranks at 180°
(Motor-car engine)
2 Sets of No. 7
No balance masses</td><td>2nd parallel
$4Aq$
$4B_2A$
23·02 lbs.</td><td>None</td><td>None</td></tr>
<tr><td>20</td><td>4 Cranks at 90°
2 Sets of No. 5
No balance masses</td><td rowspan="2">4th parallel
Aq^3
$4B_4A$
0·23 lb.</td><td>1st parallel
$2\sqrt{2}Al$
....
67·24 lb.-ft.</td><td>None</td></tr>
<tr><td>21</td><td>4 Cranks at 90°
2 Sets of No. 6
Half-balanced</td><td>1st reverse
$\sqrt{2}Al$
....
33·62 lb.-ft.</td><td>None</td></tr>
<tr><td>22</td><td>Symmetrical, 4 cranks
(Fig. 1, Chap. XI.)
$\theta_1 = 30°$, $\theta_2 = 125°$
$m_2 = 1{\cdot}50m_1$, $l_2 = 0{\cdot}41l_1$
No balance masses</td><td>4th parallel
$0{\cdot}84A_1q^3$
$3{\cdot}33B_4A_1$
0·19 lb.</td><td>2nd parallel
$1{\cdot}443Alq$
$1{\cdot}443B_2Al$
6·93 lb.-ft.</td><td>3rd
$1{\cdot}07Bq_3$
$2{\cdot}89C_3B$
0·58 lb.-ft.</td></tr>
<tr><td>23</td><td>1 Crank
4 Cylinders at 90°
Half-balanced
Impulse intervals 90°, 180°, 270°, 180°
(Fig. 8, Chap. VIII.)</td><td>None</td><td>None</td><td>None</td></tr>
<tr><td>24</td><td>2 Cranks at 90°
Cylinders at 90°
2 Sets of No. 12
Half-balanced</td><td>4th parallel
$\frac{1}{\sqrt{2}}Aq^3$
$2\sqrt{2}B_4A$
0·16 lb.</td><td>2nd perpendr.
$\sqrt{2}Alq$
$\sqrt{2}B_2Al$
6·79 lb.-ft.</td><td>1st
$2Bq$
$2C_1B$
27·32 lb.-ft.</td></tr>
<tr><td>25</td><td>2 Cranks at 180°
Cylinders at 180°
No balance masses</td><td>None</td><td>1st parallel
$2Al$
....
47·56 lb.-ft.</td><td>None</td></tr>
<tr><td>26</td><td>2 Cranks at 180°
Cylinders at 180°
2 Sets of No. 14
Half-balanced</td><td>None</td><td>1st reverse
Al
....
23·78 lb.-ft.</td><td>None</td></tr>
</table>

5	6	7	8	9
Average kinetic energy of		Largest variation of kinetic energy of		Energy ratio, $\frac{\Delta E}{R}$
Pistons.	Connecting-rods, linear.	Pistons.	Connecting-rods, linear.	
		Four Cylinders.		
		2nd, $-2·000P$ $-23·58$ ft.-lbs.	2nd, $-1·280L$ $-15·09$ ft.-lbs.	4-stroke cycle engine, 0·15–0·25
2·020P ·82 ft.-lbs.	2·723L 32·10 ft.-lbs.	4th, $-0·020P$ $-0·23$ ft.-lb.	4th, $-0·003L$ $-0·03$ ft.-lb.	single- or double-acting steam-engine, 0·02–0·04
$2·515P_1$	$3·391L_1$	3rd, $-0·293P_1$	3rd, $-0·116L_1$	double-acting steam-engine, 0·02–0·05
		4th, $-0·020P$ $-0·23$ ft.-lb.	4th, $-0·003L$ $-0·03$ ft.-lb.	4-stroke cycle engine, 0·25–0·40; single-acting steam-engine, 0·02–0·04
2·020P ·82 ft.-lbs.	2·723L 32·10 ft.-lbs.	1st, $0·202P$ 2·38 ft.-lbs. 3rd, $-0·204P$ $-2·40$ ft.-lbs.	1st, $0·080L$ 0·94 ft.-lb. 3rd, $-0·082L$ $-0·96$ ft.-lb.	4-stroke cycle engine, 0·25–0·40
		2nd, $-2·000P$ $-23·18$ ft.-lbs.	2nd, $-1·280L$ $-14·82$ ft.-lb.	4-stroke cycle engine, 0·15–0·25

	1	2	3	4
		Largest unbalanced		
No.	Type of Engine.	Force.	Longitudinal couple.	Transverse couple.
		Five Cylinders.		
27	5 *Cranks at* 120° (Fig. 18, Chap. VI.) No balance masses	6th parallel $\frac{9}{128}A^1q^5$ Where A^1 includes the total mass of the five pistons— 0·00 lbs.	None	3rd $\frac{3}{8}B^1q^3$ $C_3B_1{}^1$ 1·02 lb.-ft.
28	5 *Cranks at* 72° Equal distances between cylinders No balance masses	None	1st parallel $2{\cdot}449Al$ 10·68 lb.-ft. 2nd parallel $4{\cdot}980Alq$ 23·54 lb.-ft.	
29	5 *Cranks at* 72° (Fig. 19, Chap. VI.) No balance masses	None	2nd parallel $2{\cdot}662Al_1q$ $2{\cdot}662B_2Al_1$ 19·50 lb.-ft.	5th $\frac{75}{128}Bq^5$ $5C_5B$ 0·01 lb.-ft.
30	1 *Crank* *Cylinders at* 72° Half-balanced (Fig. 10, Chap. VIII.)	4th reverse. $\frac{5}{8}Aq^3$ $2{\cdot}5B_4A$ 0·14 lb.	None	
		Six Cylinders.		
31	6 *Cranks at* 120° Motor-car engine 2 sets of No. 15	6th parallel $\frac{27}{64}Aq^5$ $6A_6B$ 0·00 lbs.	None	3rd $\frac{9}{4}Bq^3$ $6C_3B$ 1·22 lb.-ft.
32	2 *Cranks at* 180° *Cylinders at* 60° 2 sets of No. 17 Half-balanced	None	2nd reverse $\frac{3}{2}Alq$ $1{\cdot}5B_2Ap$ 7·21 lb.-ft.	None
33	3 *Cranks at* 120° *Pairs of cylinders at* 120° 3 sets of No. 13 Half-balanced	6th parallel $\frac{27}{64}Aq^5$ $6A_6B$ 0·00 lbs.	1st reverse $\frac{\sqrt{3}}{2}Al$ .... 20·59 lb.-ft.	None

5	6	7	8	9
Average kinetic energy of		Largest variation of kinetic energy of		Energy ratio, $\frac{\Delta E}{R}$
Pistons.	Connecting-rods, linear.	Pistons.	Connecting-rods, linear.	
Five Cylinders.				
0·505P[1] 29·75 ft.-lbs. P[1] including the five pistons	0·681L[1] 40·13 ft.-lbs. L[1] including the five rods	3rd, −0·102P[1] −6·00 ft.-lbs.	3rd, −0·041L[1] −2·40 ft.-lbs.	
2·525P 29·77 ft.-lbs.	3·404L 40·13 ft.-lbs.	5th, 0·0025P 0·029 ft.-lb.	5th, 0·0010L 0·012 ft.-lb.	4-stroke cycle engines, 0·07–0·12 double-acting steam-engines, 0·000–0·003
Six Cylinders.				
3·031P 35·73 ft.-lbs.	4·085L 48·16 ft.-lbs.	3rd, −0·610P −7·19 ft.-lbs.	3rd, −0·244L −2·88 ft.-lbs.	4-stroke cycle engines, 0·05–0·08 double-acting steam-engines, 0·000–0·005
		6th, 0·0001P 0·001 ft.-lb.	6th, 0·0000L 0·0000 ft.-lbs.	
		3rd, −0·610P −7·19 ft.-lbs.	3rd, −0·244L −2·88 ft.-lbs.	

No.	1	2	3	4
		Largest unbalanced		
No.	Type of Engine.	Force.	Longitudinal couple.	Transverse couple.
	Eight Cylinders.			
34	4 *Cranks at* 180° *Cylinders at* 90° 4 sets of No. 12 No balance masses required	2nd perpendr. $4\sqrt{2}Aq$ $4\sqrt{2}B_2A$ 32·57 lbs.	None	None
35	4 *Cranks at* 90° *Cylinders at* 90° 4 sets of No. 12 Half-balanced	4th parallel $\sqrt{2}Aq^3$ $4\sqrt{2}B_4A$ 0·32 lb.	None	None
36	2 sets of No. 23 *Cranks at* 0°, 90°, *or* 180°	None	None	None
	Ten Cylinders.			
37	2 *Cranks at* 180° *Cylinders at* 36° 2 sets of No. 30 Half-balanced	None	4th reverse $\frac{5}{8}Alq^3$ $2{\cdot}5B_4Al$ 0·12 lb.-ft.	None
	Twelve Cylinders.			
38	4 *Cranks at* 90° 4 Sets of No. 18 Half-balanced Impulse intervals irregular	4th elliptical $(1 \mp \frac{1}{2})Aq^3$ $4(1 \mp \frac{1}{2})B_4A$ 0·36 to 0·12 lb.	None	None
39	3 *Cranks at* 120° 3 Sets of No. 23 Balanced Impulse intervals irregular	None	None	None
40	4 *Cranks at* 180° 4 Sets of No. 18 Half-balanced	2nd elliptical $4(1 \mp \frac{1}{2})Aq$ $4(1 \mp \frac{1}{2})B_2A$ 34·53 to 11·51 lbs.	None	None
41	6 *Cranks at* 120° *Pairs of cylinders at* 60° 6 Sets of No. 11 No balance masses required	6th parallel $\frac{27\sqrt{3}}{64}Aq^5$ $6\sqrt{3}B_6A$ 0·00 lbs.	None	None

5	6	7	8	9
Average kinetic energy of		Largest variation of kinetic energy of		Energy ratio, $\frac{\Delta E}{R}$
Pistons.	Connecting-rods, linear.	Pistons.	Connecting-rods, linear.	
Eight Cylinders.				
4·041P 47·64 ft.-lbs.	5·446L 64·24 ft.-lbs.	4th, −0·040P −0·47 ft.-lb.	4th, −0·006 −0·071 ft.-lb.	0·02–0·04
Ten Cylinders.				
5·051P 59·55 ft.-lbs.	6·808L 80·27 ft.-lbs.	10th, 0·0000P 0·000 ft.-lbs.	10th, 0·0000L 0·000 ft.-lbs.	0·015–0·030
Twelve Cylinders.				
6·061 P 71·50 ft.-lbs.	8·170 96·32 ft.-lbs.	12th, 0·0000P 0·000 ft.-lbs.	12th, 0·0000L 0·000 ft.-lbs.	0·010–0·020
		6th, 0·0001P 0·001 ft.-lb.	6th, 0·0000L 0·000 ft.-lbs.	

	1	2	3	4
		Largest unbalanced		
No.	Type of Engine.	Force.	Longitudinal couple.	Transver couple.
	Sixteen Cylinders.			
42	2 Pairs of opposed cranks at 45° 2 Sets of No. 36 Cylinders in 4 longitudinal planes	None	None	None
	Twenty-four Cylinders.			
43	*Cranks at* 120° 3 Sets of No. 36	None	None	None

5	6	7	8	9
Average kinetic energy of		Largest variation of kinetic energy of		Energy ratio, $\frac{\Delta E}{R}$
Pistons.	Connecting-rods, linear.	Pistons.	Connecting-rods, linear.	
SIXTEEN CYLINDERS.				
8·082P 95·28 ft.-lbs.	10·892L 128·48 ft.-lbs.	8th, 0·000P 0·000 ft.-lbs.	8th 0·0000L 0·000 ft.-lbs.	0·000–0·005
TWENTY-FOUR CYLINDERS.				
12·123P 142·93 ft.-lbs.	16·347L 192·72 ft.-lbs.	12th, 0·0000P 0·000 ft.-lbs.	12th, 0·0000L 0·000 ft.-lbs.	0·000–0·005

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P

THE END

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